Killing Vectors: infinitesimal symmetry transformations

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

$\varphi \to \varphi + a$ is a symmetry

$\frac{\partial}{\partial\varphi}$: vector field tangent to symmetry orbits

$(\partial_\varphi)^i = (0,1)$ in (θ,φ) coordinates.

Conserved quantities

$$S = \frac{1}{2}\int g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \, d\lambda \, , \quad \delta S = 0 \longleftrightarrow \boxed{\frac{d}{d\lambda}\left(g_{\mu\alpha}\dot{x}^\mu\right) - \frac{1}{2} g_{\mu\nu,\alpha}\dot{x}^\mu \dot{x}^\nu = 0}$$

if $g_{\mu\nu,\hat{\alpha}} = 0$ for a specific $\hat{\alpha}$, then

$$\boxed{g_{\mu\hat{\alpha}}\dot{x}^\mu = \text{constant along all geodesics.}}$$

in abstract index notation:

$$\xi = \frac{\partial}{\partial x^{\hat{\alpha}}} \; ; \quad g_{\mu\hat{\alpha}}\dot{x}^\mu = g_{\mu\nu}\xi^\nu \dot{x}^\mu = g_{ab}u^a\xi^b \, , \quad u^a = \dot{x}^a$$

now, $0 = u^c\nabla_c(\underbrace{g_{ab}u^a\xi^b}_{\text{scalar quantity defined on the geodesic}}) \longrightarrow$ is the same as $\frac{d}{d\lambda}\left(g_{\mu\hat{\alpha}}\dot{x}^\mu\right)$

$$= \underbrace{u^c(\cancel{\nabla_c g_{ab}})}_{0}\xi^b + g_{ab}\underbrace{u^c(\nabla_c u^a)}_{=0,\ \text{geodesic eqn.}}\xi^b + g_{ab}u^c u^a\nabla_c\xi^b = u^c u^a \nabla_c \xi_a$$

this is valid for any geodesic (u^a) therefore $\Longrightarrow \boxed{\nabla_{(c}\xi_{a)} = 0}$

Killing's eqn

Lemma: if $u^a u^b M_{ab} = 0 \;\forall\, u^a$, then $M_{(ab)} = 0$.

$\nabla_{(c}\xi_{a)} = \nabla_c\xi_a + \nabla_a\xi_c$

or $= \frac{1}{2}\left(\nabla_c\xi_a + \nabla_a\xi_c\right)$

depending on the convention.

Wald uses: $A_{(\mu}B_{\nu)} = \frac{1}{2!}\left(A_\mu B_\nu + A_\nu B_\mu\right)$

Number of degrees of freedom:

$$\nabla_{[a}\nabla_b\xi_{c]} = R_{[abc]}{}^d\xi_d = 0$$

$\downarrow$

$\frac{2}{3!}\left(\nabla_a\nabla_b\xi_c + \nabla_b\nabla_c\xi_a + \nabla_c\nabla_a\xi_b\right)$, because $\nabla_c\xi_b = \nabla_{[c}\xi_{b]}$.