TERMINAL CONFIGURATIONS OF STELLAR EVOLUTION

by

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I should like to thank Professor C. W. Misner for his very kind and patient assistance in the preparation of this thesis. The ideas which give direction to the discussion are primarily his. The errors which may have unobtrusively crept into the manuscript are my own.

D.L. B.

TABLE OF CONTENTS

1. Introduction 1

2. Spherical and Hyperbolic Universes 6

3. The Schwarzschild Metric 13

4. The Kruskal Extension 17

5. Some Differential Geometry 23

6. Matching Universes 31

7. Conclusions 43

8. Notes 51

9. Bibliography 52

INTRODUCTION

Certainly one of the most interesting problems of modern physics, on the frontier between general relativity and elementary particle physics, is the description of great masses of matter which have reached the endpoint of stellar evolution by exhausting all thermonuclear sources of energy. Unable to sustain any internal pressure by thermonuclear reactions, these masses of matter or cold stars must now pay attention to the gravitational curvature of space-time which causes their collapse. Although a great deal of previous work has been done on this problem, there is a conspicuous lack of agreement on the conclusions which should be drawn. J. A. Wheeler has said that one simply does not know the fate of a compact assembly of cold matter, catalyzed to the endpoint of thermonuclear evolution, when the mass of that assembly exceeds about 0.7 times the mass of the sun.[1] He goes on to point out that the problem is one of principle, having nothing directly to do with the actual course of thermonuclear evolution. If evolution of a star continues long enough, the star simply cannot eject matter, radiate photons, or emit neutrinos. It comes into the absolutely lowest state possible for a many nucleon system under the dual action of nuclear and gravitational forces. This is the state which is of interest. Therefore, he poses the question, "What is the final equilibrium state of an A-nucleon system under gravitational forces when A is large?"

But perhaps this begs the question? There might not be any equilibrium state! This was the proposal advanced by Oppenheimer and Snyder who showed in 1939 that when all thermonuclear sources of energy are exhausted a sufficiently large star will collapse, contracting asymptotically to its gravitational radius $2mG/c^2$.[2] The total time for the collapse to take place is finite for an observer comoving with the stellar matter but infinite for an observer external to the stellar matter. Light from the surface of the star is progressively reddened and can escape over a progressively narrower range of angles. Consequently, the star effectively "cuts itself off from the rest of the universe".

Certain objections have, however, been raised concerning the above analysis. J. A. Wheeler points out that[3]

(1) No mechanism of release of the gravitational energy into the surroundings is taken into account; so that this approach rules out <u>by definition</u> any approach to equilibrium if one exists. The mass of the system as viewed by a distant observer remains forever the same. In actuality the constituent particles must collide, give off heat, lose speed, and slow down their contraction.

(2) The particles are envisioned as falling into a K. Schwarzschild singularity, but this does not give an adequate representation of the forces sustained by a particle at high compression. Therefore, it appears that any answer is incomplete which does not consider the ultimate constitution of a nucleon.

(3) The particles are envisaged as "cutting themselves off from the rest of the universe". This seems to suggest that the particles lose their effect on the rest of the universe. But the discussion demands at the same time that they maintain an unchanged gravitational pull on a distant test mass - the direct opposite of losing their effect.

On the other hand, if one accepts the existence of a final equilibrium state for an A-nucleon system when A is large, one goes on to ask, what happens when another handful of nucleons is added to this equilibrium configuration. An analysis seems to indicate that the added mass must be radiated away, and in fact that a system of critical mass acts as a catalyzer, which can attract nucleons in from the outside and dissolve an equivalent number of nucleons away at the center into radiation, in order that the total number of nucleons remain below some critical number.[4] This process would be compatible with the principles of conservation of energy and conservation of momentum, but it violates conservation of nucleons.

Thus, no matter whether one believes in the lack of a final equilibrium state for an A-nucleon configuration or in its existence, certain undesirable complications arise. In this paper we shall consider the problem from Oppenheimer and Snyder's point of view that no equilibrium state need exist, so that the problem is essentially dynamic. It seems advisable at this point to reply to the criticisms against this approach raised by J. A. Wheeler. These replies seem, to the author, not really definitive, but rather tentative groping toward a more comprehensive analysis of this problem which may eventually lead to better understanding.

(1) While it is true that no mechanism of release of gravitational energy is considered, it seems quite possible that, for any solution similar to that of Oppenheimer and Snyder, the null cones of particles at the surface of this mass configuration might collapse in such a manner that very little or no energy or matter could escape. An interesting discussion of one aspect of this problem can be found

in A. P. Mills' thesis.[5] He considers the trajectories of photons emitted from the surface of a star and shows that in certain cases the photon shoots out from the star, circles around it several times, and then falls back in. It is clearly true that the energetically easiest escape for matter or energy will be along a radial geodesic. So we might expect that there would be only a small cone of directions about such radial geodesics along which matter or energy might be emitted from the star. Therefore, it seems reasonable that the amount of matter or energy which could escape from the mass configuration would be so small as to be negligible. This is also indicated by Oppenheimer and Snyder when they point out that escaping light is progressively reddened and can escape over a narrower and narrower range of angles.

(2) It has been shown that there is a limit of mass on the order of 0.7 times the mass of the sun, above which there is no equilibrium configuration for a mass of cold matter catalyzed to the endpoint of thermonuclear evolution.[6] The pressure at the center of the configuration necessarily becomes infinite as the mass is increased toward this finite limit and at the center the separation between positive and negative energy states goes to zero. The important point is that, although the details of the transition depend upon the equation of state assumed, the existence of the effects does not.[7] Consequently, in this essay a very simple equation of state will be used, which, although it neglects the forces sustained by a particle under great compression, may nevertheless lead to valuable insights into the nature of the effects themselves.

(3) This last objection seems most simply understood by an

examination of the appearance of any dynamic solution to the configuration problem in the coordinate system obtained by Kruskal in extending the Schwarzschild metric.[8] It will be seen in the sequel that the central configuration will maintain an unchanged gravitational pull on a distant test mass, although it can no longer emit any significant amount of energy which could be perceived by an exterior observer situated at the test mass. Furthermore, the exterior observer can in no way influence the central configuration after some definite critical time. But since the dynamical solution is asymptotic in nature, it would seem plausible that the gravitational effect of the central configuration would approach some limiting value different from zero in the same way that the radial coordinate approaches the gravitational radius of the configuration as limit.

Since the fate of a great number of nucleons under the action of their mutual gravitational attraction still seems very much in doubt, perhaps it would be advisable to turn at this point to the simple models which are discussed in detail in the ensuing chapters. Our results will parallel many of those obtained by Oppenheimer and Snyder although our approach and analysis will be rather different.

SPHERICAL AND HYPERBOLIC UNIVERSES

In this chapter we shall discuss universes which are isotropic and homogeneous in space at any moment of time. The properties of these universes, first studied by A. Friedmann in 1922, will be summarized rather than derived since details are available in standard texts.[9] First a reference system is chosen which moves at each point of space with the matter at that point, so that, since there is no mass flux in any direction, space will be isotropic. Then the timelike coordinate is chosen so that, at any given moment, the density of matter will be constant throughout space. These assumptions lead to a metric of the form

$$(2\text{-}1) \qquad d\tau^2 = dt^2 - dl^2$$

Since the density is constant throughout space, space is characterized completely by a single curvature parameter λ which is positive for spherical or bounded space and negative for hyperbolic or unbounded space. One may introduce spherical coordinates χ, θ, and φ in the spherical space, which may be regarded as the surface of a 4-ball of radius a as indicated in figure 2-1. Similarly one may introduce hyperbolic coordinates χ, θ, and φ for unbounded space as indicated in figure 2-2. The metrics for these spaces may then be readily computed. They are, respectively,

$$(2\text{-}2) \qquad d\tau^2 = dt^2 - a^2(t)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

$$(2\text{-}3) \qquad d\tau^2 = dt^2 - a^2(t)\left[d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

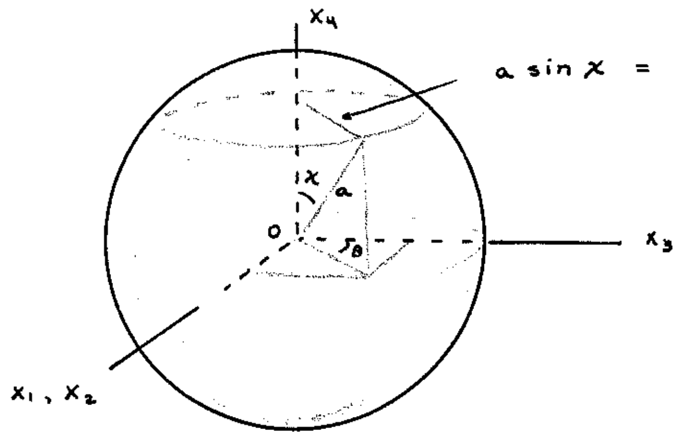
where a is, in both cases, the radius of curvature of the 3-space. The functional dependence of a upon t must be determined from the gravitational field equations and the equations of state of the appro-

FIGURE 2-1

The equation of a 3-sphere in Euclidean 4-space is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2$$

By combining the x_1 and x_2 coordinates along one axis we arrive at the following pictorial representation of a 3-sphere:



Certain other coordinate representations may be used in place of the Cartesian one, as is partially indicated above. These are:

x_1	$a \sin \chi \sin \theta \sin \varphi$	$r \sin \theta \sin \varphi$
x_2	$a \sin \chi \sin \theta \cos \varphi$	$r \sin \theta \cos \varphi$
x_3	$a \sin \chi \cos \theta$	$r \cos \theta$
x_4	$a \cos \chi$	$\sqrt{a^2 - r^2}$

The indices used to denote the various coordinates as well as the values which these coordinates may take on are indicated below:

1	χ	$[0, \pi]$
2	θ	$[0, \pi]$
3	φ	$[0, 2\pi]$

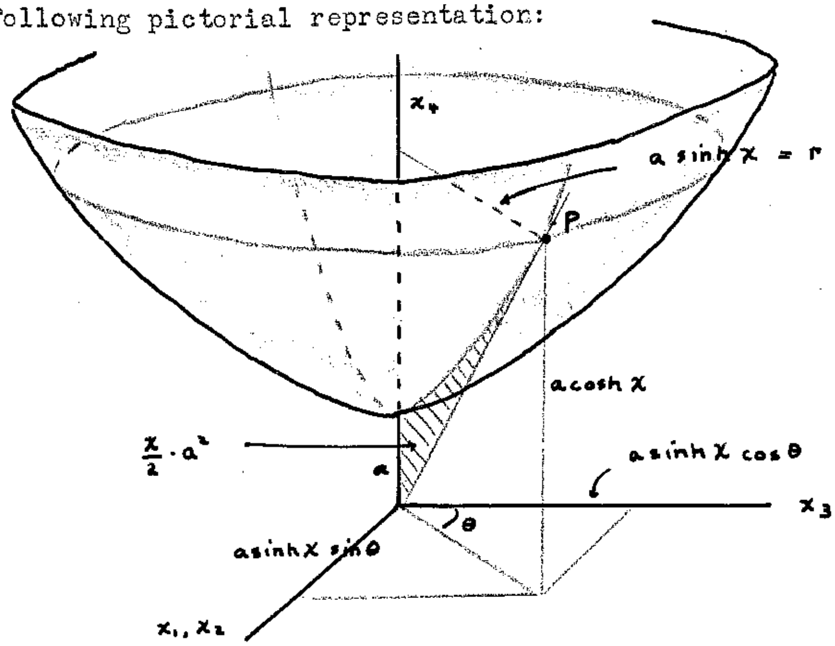
The index 0 is reserved for timelike coordinates throughout this paper.

FIGURE 2-2

The equation of a 3-pseudosphere in Minkowski 4-space is

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = -a^2$$

By combining the x_1 and x_2 coordinates along one axis we arrive at the following pictorial representation:



Certain other coordinate representations may be used in place of the Cartesian one, as is partially indicated above. These are

x_1	$a \sinh \chi \sin\theta \sin\varphi$	$r \sin\theta \sin\varphi$
x_2	$a \sinh \chi \sin\theta \cos\varphi$	$r \sin\theta \cos\varphi$
x_3	$a \sinh \chi \cos\theta$	$r \cos\theta$
x_4	$a \cosh \chi$	$\sqrt{a^2+r^2}$

The indices used to denote the various coordinates as well as the values which these coordinates may take on are indicated below:

1	χ	$[0, \infty]$
2	θ	$[0, \pi]$
3	φ	$[0, 2\pi]$

priate universe. If we assume that the universes are filled with non-interacting dust, no pressure or momentum-flux terms arise in the energy-momentum tensor T^{μ}_{ν}, whose only non-zero component is

$$(2\text{-}4) \qquad T^0_0 = \rho$$

The field equations are accordingly

$$(2\text{-}5) \qquad R^0_0 - \frac{1}{2} R = 8\pi\rho$$

Units will always be chosen so that the speed of light and the gravitational constant are unity. Making use of Dingle's formulas,[10] after some long calculations we find that for spherical universes

$$(2\text{-}6) \qquad R^0_0 - \frac{1}{2} R = \frac{3}{a^2}\left[1 + \left(\frac{da}{dt}\right)^2\right]$$

and for hyperbolic ones

$$(2\text{-}7) \qquad R^0_0 - \frac{1}{2} R = \frac{3}{a^2}\left[\left(\frac{da}{dt}\right)^2 - 1\right]$$

If we now solve (2-5) and (2-6), we obtain the following parametric relation between a and t :

$$(2\text{-}8) \qquad \begin{aligned} a &= a_0\,(1 + \cos\omega) \\ t &= a_0\,(\omega + \sin\omega) \end{aligned}$$

Similarly, (2-5) and (2-7) lead to

$$(2\text{-}9) \qquad \begin{aligned} a &= a_0\,(\cosh\omega - 1) \\ t &= a_0\,(\sinh\omega - \omega) \end{aligned}$$

In both cases,

$$(2\text{-}10) \qquad a_0 = \frac{4}{3}\pi\rho_0$$

where it has been assumed that

$$(2\text{-}11) \qquad \rho = \rho_0 / a^3$$

which we shall now justify. In our universes no matter is being created or destroyed so the total masses of the universes must remain constant. If we find the volumes of the universes as functions of t and multiply these by their densities, which we already assumed to be constant throughout the space at any particular instant t, we shall obtain their total masses. This will provide us with relations connecting ρ and a which reduce to (2-11). For reasons which will become evident in the sequel, we shall assume that the universes are spherically symmetrical but confined to the region:

(2-12) $$0 \leq \chi \leq \chi_0$$

Then the volume of a spherical universe will be

(2-13) $$v = \int_0^{\chi_0} \int_0^{\pi} \int_0^{2\pi} a^3 \sin^2\chi \sin\theta \, d\varphi \, d\theta \, d\chi$$

which becomes after integration

(2-14) $$v = 4\pi a^3 \left[\frac{\chi_0}{2} - \frac{\sin 2\chi_0}{4} \right]$$

For a hyperbolic universe

(2-15) $$v = \int_0^{\chi_0} \int_0^{\pi} \int_0^{2\pi} a^3 \sinh^2\chi \sin\theta \, d\varphi \, d\theta \, d\chi$$

and carrying out the integration

(2-16) $$v = 4\pi a^3 \left[\frac{\sinh 2\chi_0}{4} - \frac{\chi_0}{2} \right]$$

If we let M_F be the total mass of the universe in either case, then

(2-17) $$\rho = M_F / v$$

and it is clear that this equation reduces to (2-11) where for spherical universes

(2-18) $$\rho_0 = \frac{M_F}{\pi(2\chi_0 - \sin 2\chi_0)}$$

and for hyperbolic ones

(2-19) $$\rho_0 = \frac{M_F}{\pi(\sinh 2\chi_0 - 2\chi_0)}$$

Now (2-10) becomes, for the respective cases,

(2-20) $$a_0 = \frac{4M_F}{3(2\chi_0 - \sin 2\chi_0)}$$

$$(2\text{-}21) \qquad a_0 = \frac{4M_F}{3(\sinh 2\chi_0 - 2\chi_0)}$$

Returning to (2-8), we see that one cycle of expansion and contraction of the spherical universe is described by allowing the parameter ω to range from $-\pi$ to $+\pi$. Similarly in (2-9), one cycle of expansion is described by allowing ω to range from 0 to ∞. It should be noted, however, that the assumptions which led to the conclusion that the only non-zero component of T^{μ}_{ν} was T^{0}_{0} break down as a approaches zero, so the full range of the parameter may not be used.

Since dust particles in our Friedmann universes are at rest relative to space, they describe timelike geodesics along which the coordinates χ, θ, and φ are constant. Now let us introduce coordinates with radial significance in the spherical and hyperbolic universes. Referring to figures 2-1 and 2-2, we see that the proper choices of radial coordinates for these universes are respectively

$$(2\text{-}22) \qquad r = a \sin \chi$$

$$(2\text{-}23) \qquad r = a \sinh \chi$$

Then the parametric equations (2-8) and (2-9) for a as a function of t determine r as a function of t, thus determining the radial geodesics followed by a test particle whose χ, θ, and φ coordinates remain constant. The unit tangent vector field to these geodesics in either universe must be

$$(2\text{-}24) \qquad \vec{e}_0 = \frac{\vec{\partial}}{\partial t}$$

For spherical space we can choose three other vector fields to complete an orthonormal frame in the following manner:

$$(2\text{-}25) \qquad \vec{e}_1 = \frac{1}{a}\frac{\vec{\partial}}{\partial \chi}$$

$$(2\text{-}26) \qquad \vec{e}_2 = \frac{1}{a \sin \chi}\frac{\vec{\partial}}{\partial \theta}$$

(2-27) $$\vec{e}_3 = \frac{1}{a \sin \chi \sin \theta} \frac{\vec{\partial}}{\partial \varphi}$$

For hyperbolic space the corresponding vector fields are

(2-28) $$\vec{e}_1 = \frac{1}{a} \frac{\vec{\partial}}{\partial \chi}$$

(2-29) $$\vec{e}_2 = \frac{1}{a \sinh \chi} \frac{\vec{\partial}}{\partial \theta}$$

(2-30) $$\vec{e}_3 = \frac{1}{a \sinh \chi \sin \theta} \frac{\vec{\partial}}{\partial \varphi}$$

Anticipating later developments, we note that in either universe $\vec{e}_1$ is a vector field orthogonal to any manifold determined by the condition

(2-31) $$\chi = \chi_0$$

THE SCHWARZSCHILD METRIC

It is well-known that there is a coordinate system in which the metric produced by a centrally symmetric distribution of matter in the empty space surrounding this matter is static.[11] If we introduce spherical coordinates r , θ , and φ for the spacelike dimensions and t for the timelike one in the customary manner, this metric, as Schwarzschild first discovered in 1915, can be written in the following form[12]

(3-1) $$d\tau^2 = \left(1 - \frac{2M_K}{r}\right) dt^2 - \left(1 - \frac{2M_K}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

where M_K is a constant of integration which is identified with the mass of the matter in the central distribution in order that the relativistic field equations reduce to Newton's law of gravitation for large values of the radial coordinate r . It is to be noted that this metric is singular in space at $r = 2M_K$, for then the coefficient of dr^2 becomes infinite. We shall discover in the next chapter that this singularity may be removed by a suitable coordinate transformation. After the transformation the metric no longer remains static but changes with time and develops an intrinsic singularity after a finite proper time.

Now let us find the equations of motion of a test particle traveling along a radial geodesic. Since both θ and φ are constant along such a trajectory, the metric is simply

(3-2) $$d\tau^2 = X \, dt^2 - X^{-1} dr^2$$

where

(3-3) $$X = 1 - \frac{2M_K}{r}$$

The geodesics are determined by the variational principle

$$\delta \int d\tau = 0 \tag{3-4}$$

which is equivalent to

$$\delta \int \frac{d\tau^2}{d\tau^2}\, d\tau = 0 \tag{3-5}$$

Denoting differentiation with respect to τ by a dot, we may substitute (3-2) in (3-5), obtaining

$$\delta \int (X\dot{t}^2 - X^{-1}\dot{r}^2)\, d\tau = 0 \tag{3-6}$$

Since we need only one condition in addition to (3-2) to determine the radial geodesics, the simplest procedure is to vary t in (3-6). This leads to the equation

$$\int (X\dot{t}\ \delta\dot{t})\, d\tau = 0 \tag{3-7}$$

and after integrating by parts we have

$$\int \left[\frac{d}{d\tau}(X\dot{t})\ \delta t \right] d\tau = 0 \tag{3-8}$$

Therefore,

$$X\dot{t} = \alpha \tag{3-9}$$

where α is a constant of integration analogous to the energy of the test particle. It will be called the radial geodesic energy parameter. We may now write (3-2) as

$$X\dot{t}^2 - X^{-1}\dot{r}^2 = 1 \tag{3-10}$$

and use (3-9) to eliminate $\dot{t}$, which gives

$$\dot{r} = \pm\sqrt{\alpha^2 - X} \tag{3-11}$$

Manasse obtained the following parametric solution for (3-11), assuming $\alpha < 1$ [13]:

$$\begin{aligned} r &= \frac{M_K}{1-\alpha^2}(1 + \cos\omega) \\ \tau &= \frac{M_K}{(1-\alpha^2)^{3/2}}(\omega + \sin\omega) \end{aligned} \tag{3-12}$$

On the other hand, if $\alpha > 1$, the solution is

$$(3\text{-}13)\qquad \begin{aligned} r &= \frac{M\kappa}{\alpha^2 - 1}(\cosh\omega - 1) \\ \tau &= \frac{M\kappa}{(\alpha^2-1)^{3/2}}(\sinh\omega - \omega) \end{aligned}$$

We can set up an orthonormal frame at each point of the radial geodesics by using the tangent vector fields $\vec{e}_0$ to these geodesics, the unit vector fields $\vec{e}_2$ and $\vec{e}_3$ along the θ and φ coordinate gradients, and a fourth vector field $\vec{e}_1$ determined by the three already chosen. It is immaterial whether the energy parameter α is greater than or less than unity. The required vector fields are:

$$(3\text{-}14)\qquad \vec{e}_0 = \dot{t}\,\frac{\vec{\partial}}{\partial t} + \dot{r}\,\frac{\vec{\partial}}{\partial r}$$

$$(3\text{-}15)\qquad \vec{e}_2 = \frac{1}{r}\,\frac{\vec{\partial}}{\partial\theta}$$

$$(3\text{-}16)\qquad \vec{e}_3 = \frac{1}{r\sin\theta}\,\frac{\vec{\partial}}{\partial\varphi}$$

This leaves

$$(3\text{-}17)\qquad \vec{e}_1 = \rho\,\frac{\vec{\partial}}{\partial t} + \gamma\,\frac{\vec{\partial}}{\partial r}$$

where ρ and γ are to be determined from the orthonormality requirements:

$$(3\text{-}18)\qquad X\rho^2 - X^{-1}\gamma^2 = -1$$

$$(3\text{-}19)\qquad X\dot{t}\rho - X^{-1}\dot{r}\gamma = 0$$

These may be solved readily to show that

$$(3\text{-}20)\qquad \vec{e}_1 = X^{-1}\dot{r}\,\frac{\vec{\partial}}{\partial t} + X\dot{t}\,\frac{\vec{\partial}}{\partial r}$$

It is to be noted that $\vec{e}_1$ is orthogonal to the manifold generated by a particle free to move along radial geodesics and θ and φ

coordinate curves. Also (3-14) and (3-20) may be rewritten using (3-9) and (3-11) in the following manner:

$$\vec{e}_0 = \frac{\alpha}{X}\frac{\vec{\partial}}{\partial t} \pm \sqrt{\alpha^2 - X}\,\frac{\vec{\partial}}{\partial r} \tag{3-21}$$

$$\vec{e}_1 = \pm\frac{\sqrt{\alpha^2 - X}}{X}\frac{\vec{\partial}}{\partial t} + \alpha\,\frac{\vec{\partial}}{\partial r} \tag{3-22}$$

THE KRUSKAL EXTENSION

We shall now investigate the singularity in the metric (3-1), which we recall was

$$(4\text{-}1) \qquad d\tau^2 = \left(1 - \frac{2M_K}{r}\right) dt^2 - \left(1 - \frac{2M_K}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

It is apparent that the coefficient of dr^2 becomes infinite at $r = 2M_K$, but it has long been known that this is due only to the choice of coordinates.[14] The proof of this statement may be based upon the fact that the curvature invariants are not singular at $r = 2M_K$[15], but it is more convincing to note that the geodesic equations determined by (4-1) show a singular behavior only at $r = 0$[16]. The source of the difficulty in (4-1) is that the radial null-cones collapse at $r = 2M_K$, as may most easily be seen by finding the tangent vector fields to the null geodesics. Along radial null-cones

$$(4\text{-}2) \qquad 0 = \left(1 - \frac{2M_K}{r}\right) dt^2 - \left(1 - \frac{2M_K}{r}\right)^{-1} dr^2$$

or

$$(4\text{-}3) \qquad dt = \pm \left(1 - \frac{2M_K}{r}\right)^{-1} dr$$

so the tangent vector fields $\vec{e}_+$ and $\vec{e}_-$ are

$$(4\text{-}4) \qquad \vec{e}_\pm = \frac{\vec{\partial}}{\partial t} \pm \left(1 - \frac{2M_K}{r}\right) \frac{\vec{\partial}}{\partial r}$$

Now as r tends to $2M_K$, $\vec{e}_+$ and $\vec{e}_-$ collapse into $\frac{\vec{\partial}}{\partial t}$; consequently, the null-cones collapse. This suggests that a transformation to a spherically symmetric coordinate system in which radial light rays everywhere have slope ± 1 will prevent the null-cones from collapsing and, for that reason, will eliminate the singularity in space at $r = 2M_K$. Therefore, we look for a transformation of (t , r) into (v , u) such that $\vec{e}_+$ and $\vec{e}_-$ in the new coordinates will be proportional to the vectors $\frac{\vec{\partial}}{\partial v} \pm \frac{\vec{\partial}}{\partial u}$ respectively, which means that (4-1) will become

$$(4\text{-}5) \quad d\tau^2 = f^2(dv^2 - du^2) - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Kruskal has found that if we require f to depend upon r alone and to remain finite and non-zero for $u = v = 0$, the following essentially unique equations of transformation between the exterior Schwarzschild coordinates $r > 2M_K$, and the quadrant $u > |v|$ in the plane of the new coordinates are determined:[17]

$$(4\text{-}6) \quad u = \sqrt{\frac{r}{2M_K} - 1}\ \exp\left(\frac{r}{4M_K}\right) \cosh\left(\frac{t}{4M_K}\right)$$

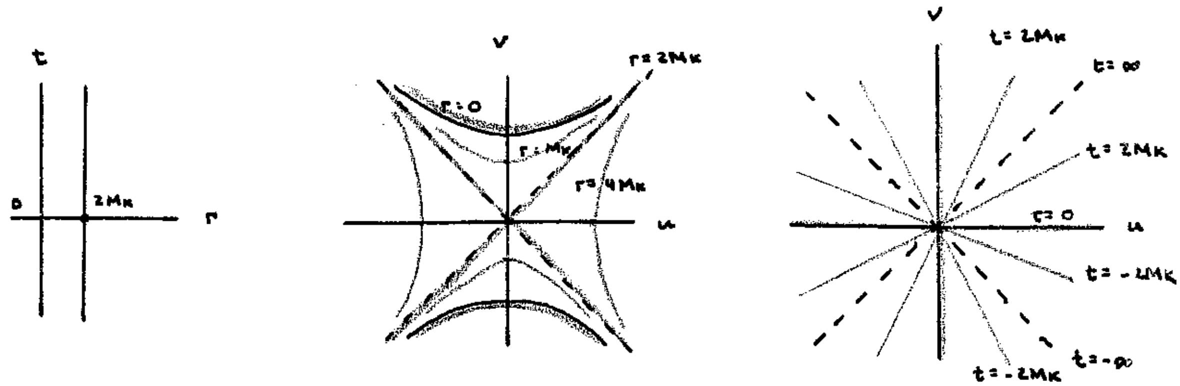
$$(4\text{-}7) \quad v = \sqrt{\frac{r}{2M_K} - 1}\ \exp\left(\frac{r}{4M_K}\right) \sinh\left(\frac{t}{4M_K}\right)$$

The new coordinates give an analytic extension of the limited region of spacetime which is described without singularity by the Schwarzschild coordinates. The metric of the extension joins smoothly without singularity to the metric at the boundary $r = 2M_K$. As Kruskal goes on to point out, this extension is <u>maximal</u> in the sense that now every geodesic either runs into the barrier of intrinsic singularities at $r = 0$ or is continuable out to infinity. The extension has a topology $S^2 \times R^2$ and constitutes a "bridge" between two Euclidean spaces as was earlier sought by Einstein and Rosen.[18] A diagram showing corresponding regions of the (t, r) and (v, u) planes has been reproduced from Kruskal's paper in figure 4-1. From the form of the transformation f can be found, and the new metric becomes

$$(4\text{-}8) \quad d\tau^2 = \frac{32 M_K^3}{r} \exp\left(\frac{-r}{2M_K}\right)(dv^2 - du^2) - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Although r has lost its interpretation as radial displacement, it

FIGURE 4-1



In the region $u > |v|$

$$u = \sqrt{\frac{r}{2Mk} - 1}\ \exp\left(\frac{r}{4Mk}\right)\cosh\left(\frac{t}{4Mk}\right) \qquad \left(\frac{r}{2Mk} - 1\right)\exp\frac{r}{2Mk} = u^2 - v^2$$

$$v = \sqrt{\frac{r}{2Mk} - 1}\ \exp\left(\frac{r}{4Mk}\right)\cosh\left(\frac{t}{4Mk}\right) \qquad \frac{t}{4Mk} = \operatorname{arctanh}\frac{v}{u}$$

$$f^2 = \frac{32Mk^3}{r}\exp\left(\frac{-r}{2Mk}\right)$$

In the (u , v) plane, curves of constant r are hyperbolas asymptotic to the lines $r = 2Mk$ while t is constant on straight lines through the origin. The region $r > 2Mk$ corresponds to the region $u > |v|$. The metric is entirely regular in the entire region between the two branches of the hyperbola $r = 0$. The radial null geodesics are lines with slope ± 1 in the (u , v) plane. Note that if a particle crosses the $r = 2Mk$ line into the interior, it can never get back out but must inevitably hit the irremovable singularity at $r = 0$. To obtain formulas valid in other regions, the following replacements are to be made:

$u > \|v\|$	$-u > \|v\|$	$v > \|u\|$	$-v > \|u\|$
u	$-u$	v	$-v$
v	$-v$	u	$-u$
$\frac{r}{2Mk} - 1$	$\frac{r}{2Mk} - 1$	$1 - \frac{r}{2Mk}$	$1 - \frac{r}{2Mk}$

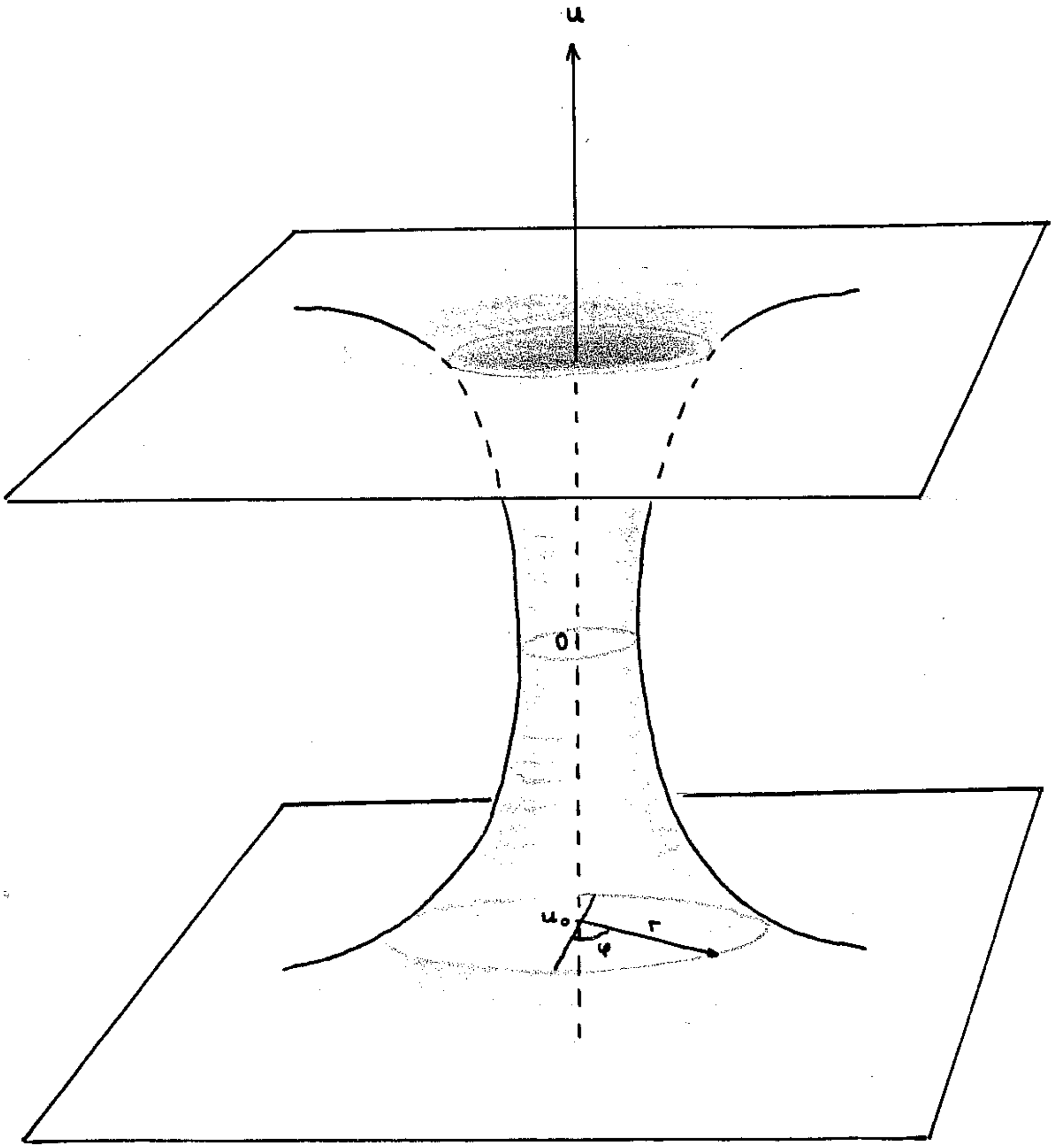
still provides a measure of space in the sense that if we choose any spacelike surface determined by $v = v_0$ and at each u calculate the circumference of the $\theta = \frac{\pi}{2}$ section of this surface, we find that

(4-9) circumference $= \int ds = \int_0^{2\pi} r\,d\varphi = 2\pi r$

Consequently, as one moves along the surface toward $u = 0$, the circumference decreases to a minimum value. Figure 4-2 shows this $\theta = \frac{\pi}{2}$ section, the bridge between two Euclidean spaces. It is also clear from figures 4-1 and 4-2 that, as v increases, the minimum circumference of the bridge will decrese to zero and the two Euclidean spaces will separate from one another. Thus, a metric previously regarded as singular in space and regular in time is more properly regarded as regular in space but intrinsically singular after some finite proper time! Paradoxically, however, no signal from any point whose coordinates satisfy $v > |u|$ can ever reach an observer located in the region $r > 2M_K$, so this exterior observer will never perceive the singularity at $r = 0$, not even in an infinite proper time!

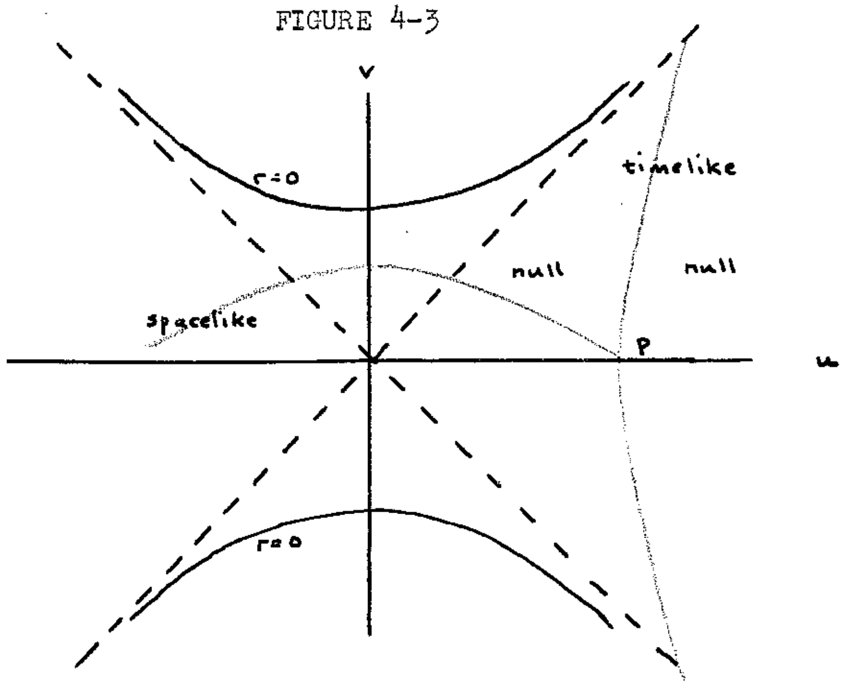
As was remarked earlier, the equations for geodesics are singular only at $r = 0$, so the radial geodesics which were determined in the previous chapter (3-30) may be continued into the extension. Some of these geodesics have actually been calculated and plotted by R. Fuller and J. A. Wheeler.[19] Their results are indicated in figure 4-3. The symmetry of timelike geodesics about the v-axis is to be noted.

FIGURE 4-2



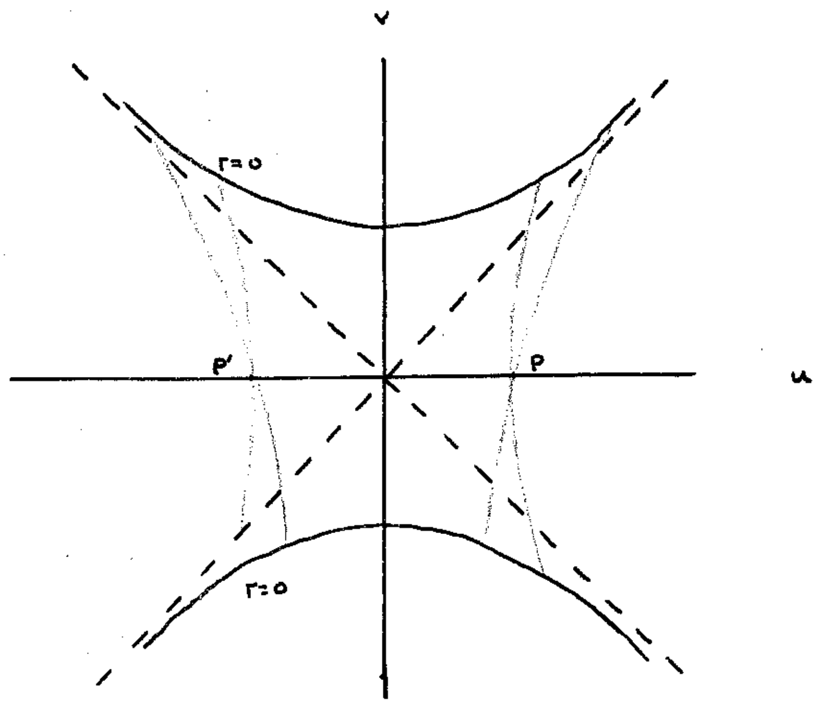
A bridge connecting two Euclidean spaces

FIGURE 4-3



Radial geodesics in Kruskal coordinates

Symmetry of timelike geodesics about v-axis



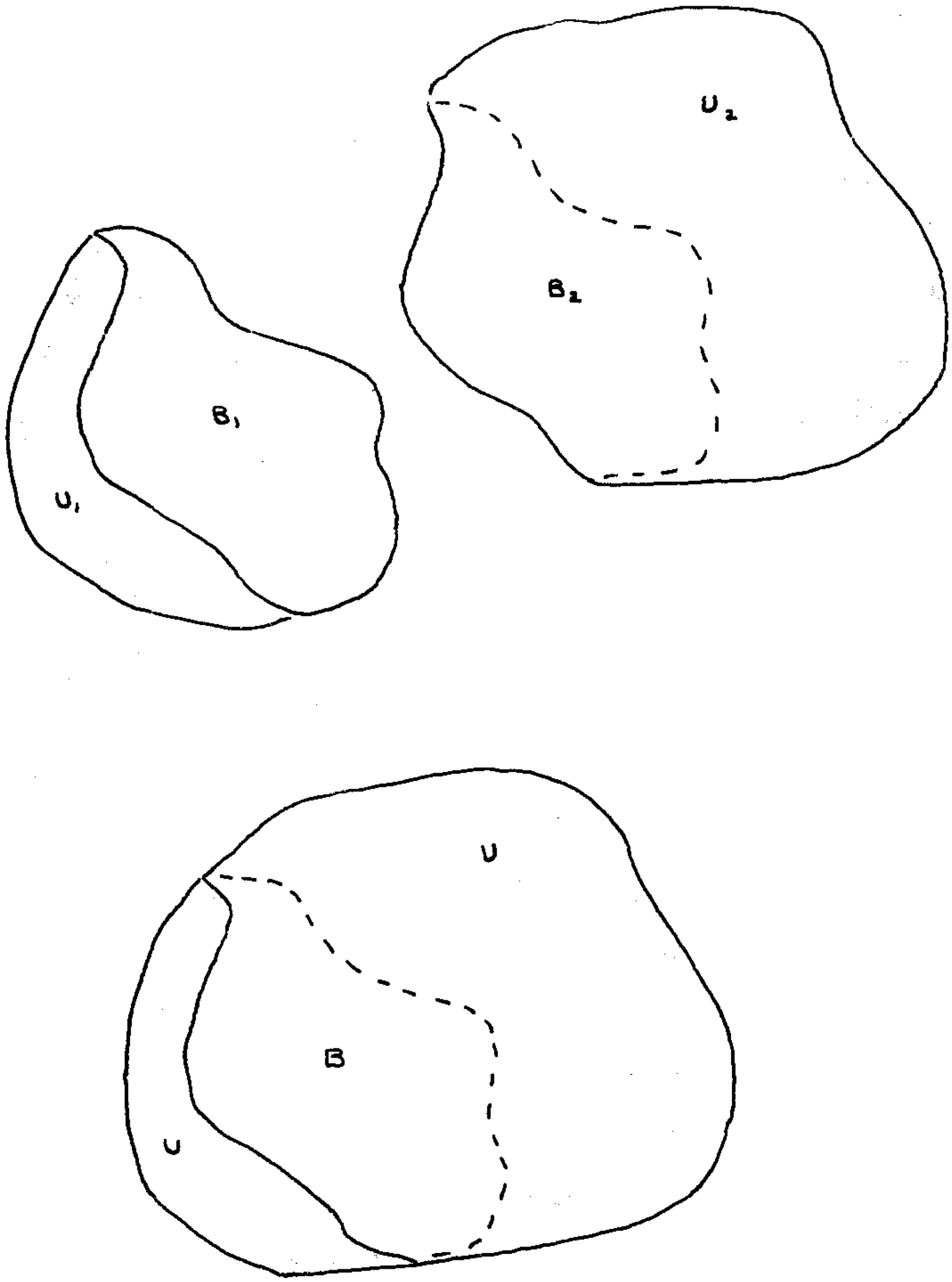
SOME DIFFERENTIAL GEOMETRY

Before plunging ahead with theoretical considerations, it would be advisable to highlight the ideas involved by a simple model. Imagine a ping-pong ball blown up to a grand scale so that its radius is the mean radius of the orbit of the earth and let the center of the ball coincide with the center of the sun. The surface of this ball is then a sphere which seperates space beyond the earth's orbit from space within that orbit. If we now leave the ping-pong ball in place for some time, it is clear that the surface of the ball will generate a certain three-dimensional manifold in spacetime. "Within" this manifold is a four-dimensional manifold generated by space interior to the orbit of the earth, and "without" is the four-dimensional manifold generated by space beyond the earth's orbit. The words "within" and "without" may be misleading. What is meant is simply that any spacetime geodesic curve which connects a point "within" to one "without" must have at least one point in common with the manifold generated by the surface of the ball. Now suppose that interior spacetime has been separated from exterior spacetime, much as if one had cut an orange in half and then physically pulled the halves apart. The manifold generated by the surface of the ping-pong ball now serves as the entire boundary for both interior and exterior spacetime, but, since the spacetimes have been separated, there are two identical copies of the boundary manifold, one for each spacetime. Using the orange as an example again, the assertion is simply that one slice produces two boundaries, the right and the left, which are necesarily identical in size and shape. This is indicated pictorially in figure

5-1, in which, at the bottom, the orange U with slice B is shown, and, at the top, the right half has been separated from the left half so that B_1 and B_2 are identical copies of B. Evidently, this whole process may be reversed, for we need only begin at the top in figure 5-1 by fitting U_1 and U_2 together by matching B_1 to B_2, whereupon we once again have a whole orange. But, since we are more interested in universes than oranges, as the notation in the figure probably indicates, we return to our ping-pong ball universes to note that the exterior and interior spacetimes, which we separated earlier, can be put back together merely by fitting their boundary manifolds together.

Now our problem can be formulated quite succinctly. Given two universes U_1 and U_2 with boundary manifolds B_1 and B_2, what conditions must be satisfied in order that U_1 and U_2 can be fitted together to form a single universe U by matching B_1 to B_2? More particularly, can a Friedmann dust-filled universe be regarded, at any given moment, as an ordinary star embedded in an otherwise empty space, that is, in a Kruskal universe? The Friedmann universe, at a given instant, would then correspond to the interior of the ping-pong ball which was discussed before, but the surface of the universe at this instant and, consequently, the manifold generated by the surface in spacetime would still need to be defined. Analogously, the Kruskal universe would correspond to spacetime outside the manifold generated by the surface of the ping-pong ball, and a boundary manifold for this universe would also remain to be defined. The remainder of this chapter will be devoted to answering the first of these questions, namely, that of finding conditions on B_1, U_1, B_2,

FIGURE 5-1



and U_2 allowing U_1 and U_2 to be joined , and to simplifying these conditions to fit the special cases which will occur in matching Friedmann and Kruskal universes. The next chapter will show how the boundary manifolds for the Friedmann and Kruskal universes must be chosen to satisfy the conditions for matching.

Although our results can be generalized to Riemannian manifolds of higher dimension, we will henceforth suppose that U_1 and U_2 are four-dimensional and that B_1 and B_2 are three-dimensional. Greek indices will range on 0,1,2,3; Latin indices, on 1,2,3. We further suppose that coordinates x_1^μ and x_2^μ have been introduced in U_1 and U_2 and that the manifolds B_1 and B_2 are determined parametrically by

$$
(5\text{-}1)\qquad
\begin{aligned}
x_1^\mu &= x_1^\mu\,(\bar{x}_1^1,\ \bar{x}_1^2,\ \bar{x}_1^3) \\
x_2^\mu &= x_2^\mu\,(\bar{x}_2^1,\ \bar{x}_2^2,\ \bar{x}_2^3)
\end{aligned}
$$

Also we suppose that the manifolds B_1 and B_2 are to be identified by requiring that

$$
(5\text{-}2)\qquad \bar{x}_1^i \quad\longleftrightarrow\quad \bar{x}_2^i
$$

In each space the choice of coordinates determines the metric

$$
(5\text{-}3)\qquad
\begin{aligned}
d\tau_1^2 &= {}_1g_{\mu\nu}\ dx_1^\mu\ dx_1^\nu \\
d\tau_2^2 &= {}_2g_{\mu\nu}\ dx_2^\mu\ dx_2^\nu
\end{aligned}
$$

A boundary condition which seems physically reasonable is to require that, after B_1 and B_2 are identified and a coordinate system has been chosen for U, the coefficients of the metric tensor of U in this coordinate system and their normal derivatives with respect to M should be continuous across B. Let this boundary condition be accepted. We shall show that, to ensure its fulfillment, we need only require that the first and second fundamental forms of B_1 and B_2 be identical, if B_1 corresponds to B_2 via (5-2).

The first fundamental forms of B_1 and B_2 will be denoted by I_1 and I_2; the second fundamental forms, by II_1 and II_2. Then, by definition:[20]

$$(5\text{-}4)\qquad \begin{aligned} I_1 &= {}_1\bar{g}_{ij}\, d\bar{x}_1^i\, d\bar{x}_1^j \\ I_2 &= {}_2\bar{g}_{ij}\, d\bar{x}_2^i\, d\bar{x}_2^j \end{aligned}$$

where

$$(5\text{-}5)\qquad \begin{aligned} {}_1\bar{g}_{ij} &= {}_1g_{\mu\nu} \frac{\partial x_1^\mu}{\partial \bar{x}_1^i} \frac{\partial x_1^\nu}{\partial \bar{x}_1^j} \\ {}_2\bar{g}_{ij} &= {}_2g_{\mu\nu} \frac{\partial x_2^\mu}{\partial \bar{x}_2^i} \frac{\partial x_2^\nu}{\partial \bar{x}_2^j} \end{aligned}$$

Next we will assume that $\det({}_1\bar{g}_{ij}) \neq 0$ and $\det({}_2\bar{g}_{ij}) \neq 0$, since this ensures the existence of non-null vector fields in U_1 and U_2 orthogonal respectively to B_1 and B_2.[21] Denote these vector fields by ξ_1^μ and ξ_2^μ. In our applications these vectors will be spacelike unit vectors, so we shall make that assumption here as well. In terms of ξ_1^μ and ξ_2^μ, II_1 and II_2 are defined by:[22]

$$(5\text{-}6)\qquad \begin{aligned} II_1 &= {}_1\Omega_{ij}\, d\bar{x}_1^i\, d\bar{x}_1^j \\ II_2 &= {}_2\Omega_{ij}\, d\bar{x}_2^i\, d\bar{x}_2^j \end{aligned}$$

where

$$(5\text{-}7)\qquad \begin{aligned} {}_1\Omega_{ij} &= -(\xi_1)_{\mu;\nu} \frac{\partial x_1^\mu}{\partial \bar{x}_1^i} \frac{\partial x_1^\nu}{\partial \bar{x}_1^j} \\ {}_2\Omega_{ij} &= -(\xi_2)_{\mu;\nu} \frac{\partial x_2^\mu}{\partial \bar{x}_2^i} \frac{\partial x_2^\nu}{\partial \bar{x}_2^j} \end{aligned}$$

In these relations $\xi_{\mu;\nu}$ stands for the covariant derivative of ξ_μ with respect to x^ν and the relevant $g_{\mu\nu}$. Let us now assume that, after the identification (5-2) has been made, ${}_1\bar{g}_{ij} = {}_2\bar{g}_{ij}$ and that ${}_1\Omega_{ij} = {}_2\Omega_{ij}$. Drop the subscripts so now for B in U

$$(5\text{-}8)\qquad \begin{aligned} \bar{x}^i &= \bar{x}_1^i = \bar{x}_2^i \\ \bar{g}_{ij} &= {}_1\bar{g}_{ij} = {}_2\bar{g}_{ij} \\ \Omega_{ij} &= {}_1\Omega_{ij} = {}_2\Omega_{ij} \\ \xi^\mu &= \xi_1^\mu = \xi_2^\mu \end{aligned}$$

Introduce $\bar{x}^i$ as coordinates x^i in U and add a fourth coordinate x^0 as follows. In a sufficiently small neighborhood of B we can construct geodesics orthogonal to B so that each point $\bar{x}$ in B uniquely determines a single geodesic. Let x^0 be the spacelike interval measured along this geodesic from the point $\bar{x}$ in B. Then the coordinates for U are:

$$
(5\text{-}9)\qquad
\begin{aligned}
x^0 &= s \qquad (dx^1 = dx^2 = dx^3 = 0)\\
x^1 &= \bar{x}^1\\
x^2 &= \bar{x}^2\\
x^3 &= \bar{x}^3
\end{aligned}
$$

It follows that the interval in U is given by

$$(5\text{-}10)\qquad d\tau^2 = -(dx^0)^2 + \bar{g}_{ij}\, dx^i\, dx^j$$

and that

$$(5\text{-}11)\qquad \xi^\mu = \xi_\mu = \frac{\vec{\partial}}{\partial x^0}$$

$$(5\text{-}12)\qquad \frac{\partial x^\mu}{\partial \bar{x}^i} = \delta^\mu_i \qquad \frac{\partial x^\nu}{\partial \bar{x}^j} = \delta^\nu_j$$

The coefficients of the second fundamental form II for B are:

$$(5\text{-}13)\qquad \Omega_{ij} = -\xi_{\mu;\nu}\,\frac{\partial x^\mu}{\partial \bar{x}^i}\,\frac{\partial x^\nu}{\partial \bar{x}^j}$$

and, by definition,

$$(5\text{-}14)\qquad \xi_{\mu;\nu} = \frac{\partial \xi_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu}\,\xi_\lambda$$

If we combine relations (5-11) to (5-14), we find that

$$(5\text{-}15)\qquad \Omega_{ij} = \Gamma^0_{\mu\nu}\,\delta^\mu_i\,\delta^\nu_j = \Gamma^0_{ij}$$

From (5-10) it is clear that

$$(5\text{-}16)\qquad g^{0\nu} = \delta^{0\nu}$$

and, consequently,

$$(5\text{-}17) \qquad \Omega_{ij} = \Gamma^{0}_{ij} = \Gamma_{ij,0} = -\frac{1}{2}\frac{\partial \bar{g}_{ij}}{\partial x^{0}}$$

Therefore,

$$(5\text{-}18) \qquad \frac{\partial g_{ij}}{\partial x^{0}} = \frac{\partial \bar{g}_{ij}}{\partial x^{0}} = -2\Omega_{ij}$$

This shows that the coefficients $g_{\mu\nu}$ are continuous across B and that the normal derivative, that is, the derivative with respect to x^{0}, of $g_{\mu\nu}$ exists across B. Continuity of this derivative follows from the fact that the above reasoning actually shows that the derivatives from the left and from the right exist and are equal.

Let us now turn to considerations which will simplify (5-5) and (5-7), thereby eliminating unnecessary calculations in the next chapter. Our coordinates x_1^{μ}, x_2^{μ}, $\bar{x}_1^{i}$, $\bar{x}_2^{i}$ will always be orthogonal, which is to say that the off-diagonal terms in the metric tensors ${}_1g_{\mu\nu}$, ${}_2g_{\mu\nu}$, ${}_1\bar{g}_{ij}$, ${}_2\bar{g}_{ij}$ will always be zero. We shall always make the identifications

$$(5\text{-}19) \qquad \begin{array}{llll} x_1^{2} = \bar{x}_1^{2} & & x_2^{2} = \bar{x}_2^{2} \\ x_1^{3} = \bar{x}_1^{3} & & x_2^{3} = \bar{x}_2^{3} \end{array}$$

in order to simplify the partial derivatives appearing in (5-5) and (5-7). Finally we note the following result:[23] If $P(x^{i})$ and $P'(x^{i}+dx^{i})$ are nearby points of a hypersurface V_n and if C is the geodesic in V_n determined by these points, it follows that p, given by

$$(5\text{-}20) \qquad 2p = \Omega_{ij}\, dx^{i}\, dx^{j}$$

is the distance from P' to the geodesic of V_{n+1} tangent to C at P. Specializing to our case, if the $\bar{x}^{1}$ coordinate curves in B_1 and B_2 are chosen to be geodesics of U_1 and U_2 respectively, then p in

(5-20) must be zero and it follows that

$$(5\text{-}21) \qquad \begin{aligned} {}_1\Omega_{11} &= 0 \\ {}_2\Omega_{11} &= 0 \end{aligned}$$

MATCHING UNIVERSES

In this chapter we will show how the boundary manifolds for the Friedmann and Kruskal universes must be chosen in order that, after coordinate identifications as in (5-2) have been made, their first and second fundamental forms will be identical. First we shall consider the case of a spherical Friedmann universe and a Kruskal universe whose radial geodesics are characterized by an energy parameter α less than unity. Then we shall repeat the calculations for a hyperbolic Friedmann universe and a Kruskal universe in which the parameter α is greater than unity.

Since Friedmann and Kruskal universes are spherically symmetric, we expect our matching problem to be quite similar to the ping-pong ball analogue which was discussed earlier. Specifically, we must define a time with the same intrinsic significance in both universes, and then at any instant of this time we should be able to identify radial coordinates in both universes in such a way that the spherical surfaces ${}_F S$ and ${}_K S$ determined by constant time and radius in the respective universes will fit together properly. If we suppose that the matching is possible, we see immediately how to proceed. If the universes are matched along ${}_F S$ and ${}_K S$ at one particular moment, then particles belonging to ${}_F S$ and ${}_K S$ which are to be identified must behave in the same way. We have already seen that all particles in the Friedmann universe move along radial geodesics. Hence the counterparts in ${}_K S$ of particles in ${}_F S$ must follow radial geodesics in the Kruskal universe also. This leads to an obvious choice for the time parameter, having the same significance in either universe - namely, proper time along radial geodesics! Referring to our earlier results (2-8) and (3-12), we recall that the parametric equations for radial geodesics were:

$$
\begin{aligned}
&{}_F U: \quad && a = a_0 \, (1 + \cos \omega_F) \\
& && \tau_F = t = a_0 \, (\omega_F + \sin \omega_F) \\
(6\text{-}1) \quad &{}_K U: && r = \frac{M_K}{1-\alpha^2} \, (1 + \cos \omega_K) \\
& && \tau_K = \frac{M_K}{(1-\alpha^2)^{3/2}} \, (\omega_K + \sin \omega_K)
\end{aligned}
$$

In order to identify τ_F and τ_K we must require that

(6-2) $$\omega_F = \omega_K = \omega$$

and

(6-3) $$a_0 = \frac{M_K}{(1-\alpha^2)^{3/2}}$$

Equation (6-3) will be discussed in detail in the next chapter; meanwhile, we will concentrate exclusively on the matching problem. If our universes, which have been matched by identifying ${}_F S$ and ${}_K S$ at some time τ_0 , are to remain matched as the time parameter varies, the manifolds ${}_F B$ and ${}_K B$ generated by ${}_F S$ and ${}_K S$ must have identical first and second fundamental forms, according to the discussion of the last chapter. The equations of these manifolds are:

(6-4) ${}_F B$:
$$
\begin{aligned}
t &= a_0 \, (\omega + \sin \omega) \\
\chi &= \chi_0 \\
\theta &= \theta \\
\varphi &= \varphi
\end{aligned}
$$

(6-5) ${}_K B$:
$$
\begin{aligned}
t &= t(\omega) \\
r &= \frac{M_K}{1-\alpha^2} \, (1 + \cos \omega) \\
\theta &= \theta \\
\varphi &= \varphi
\end{aligned}
$$

In (6-5), $t = t(\omega)$ must be determined from (3-9) and (3-12) which are reproduced below:

$$(6\text{-}6)\qquad \begin{aligned} \tau &= \frac{M\kappa}{(1-\alpha^2)^{3/2}}\,(\omega + \sin\omega) \\ \dot{t} &= \frac{dt}{d\tau} = \frac{\alpha}{X} \end{aligned}$$

We shall adopt the following notation when utilizing results from the last chapter:

$$(6\text{-}7)\qquad {}_F U: \quad x^0 = t, \quad x^1 = \chi, \quad x^2 = \theta, \quad x^3 = \varphi \qquad\qquad \bar{x}^1 = \omega, \quad \bar{x}^2 = \theta, \quad \bar{x}^3 = \varphi$$

$$(6\text{-}8)\qquad {}_K U: \quad x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi \qquad\qquad \bar{x}^1 = \omega, \quad \bar{x}^2 = \theta, \quad \bar{x}^3 = \varphi$$

Recalling (2-2) and (3-1),

$$(6\text{-}9)\qquad \begin{aligned} {}_F U: \quad d\tau^2 &= dt^2 - a^2(t)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \\ {}_K U: \quad d\tau^2 &= X\,dt^2 - X^{-1}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \end{aligned}$$

Now combining (5-4), (5-5), (6-4), and (6-5) we find that

$$(6\text{-}10)\qquad \begin{aligned} {}_F I &= a^2\, d\omega^2 - a^2 \sin^2\chi_0\,(d\theta^2 + \sin^2\theta\, d\varphi^2) \\ {}_K I &= \frac{r^2}{1-\alpha^2}\, d\omega^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \end{aligned}$$

are the first fundamental forms induced upon ${}_F B$ and ${}_K B$ by (6-9). We now identify the coordinates ω, θ, and φ in ${}_F B$ and ${}_K B$. Since ${}_F I$ and ${}_K I$ must be identical after this identification, the following conditions must be satisfied:

$$(6\text{-}11)\qquad r^2 = a^2 \sin^2\chi_0$$

$$(6\text{-}12)\qquad \frac{r^2}{1-\alpha^2} = a^2$$

We may conclude that

(6-13) $$\sin^2 \chi_0 = 1 - \alpha^2$$

or

(6-14) $$\cos \chi_0 = \pm \alpha$$

This gives us one condition from the two equations (6-11) and (6-12). We take the other condition to be

(6-15) $$r = a \sin \chi_0$$

which is automatically satisfied if (6-2), (6-3), and (6-13) are.

Now we will verify that if the conditions (6-14) and (6-15) are satisfied the second fundamental forms $_F$II and $_K$II of $_F$B and $_K$B are identical. From (2-25) and (3-22) we know that the vector fields which are orthogonal to $_F$B and $_K$B are:

(6-16) $$\begin{aligned} {}_F\xi &= \frac{1}{a} \frac{\vec{\partial}}{\partial \chi} \\ {}_K\xi &= \pm \frac{\sqrt{\alpha^2 - x}}{x} \frac{\vec{\partial}}{\partial t} + \alpha \frac{\vec{\partial}}{\partial r} \end{aligned}$$

This allows us to evaluate the coefficients of the second fundamental forms, $_F\Omega_{ij}$ and $_K\Omega_{ij}$. We note that, as mentioned at the end of the last chapter,

(6-17) $${}_F\Omega_{11} = {}_K\Omega_{11} = 0$$

since the ω -coordinate curves in $_F$B and $_K$B are radial geodesics in their respective universes. For the other components, from (5-13) and (5-14) we have

(6-18) $$\Omega_{ij} = -\left(\frac{\partial \xi_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} \xi_\lambda \right) \frac{\partial x^\mu}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^j}$$

For all Friedmann coefficients, (6-16) allows us to rewrite (6-18) as

(6-19) $${}_F\Omega_{ij} = -\frac{\partial \xi_1}{\partial x^\nu} \frac{\partial x^1}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^j} + \Gamma^1_{\mu\nu} \xi_1 \frac{\partial x^\mu}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^j}$$

From (6-4),

$$(6\text{-}20) \qquad \frac{\partial x^1}{\partial \bar{x}^i} = 0$$

so (6-19) becomes

$$(6\text{-}21) \qquad {}_F\Omega_{ij} = \xi_1 \Gamma^1_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^j}$$

Then, using (6-4) and (6-9), we obtain

$$(6\text{-}22) \qquad \begin{aligned} {}_F\Omega_{12} &= \xi_1 \Gamma^1_{02} \frac{dt}{d\omega} = \xi^1 \Gamma_{02,1} \frac{dt}{d\omega} = 0 \\ {}_F\Omega_{13} &= \xi_1 \Gamma^1_{03} \frac{dt}{d\omega} = \xi^1 \Gamma_{03,1} \frac{dt}{d\omega} = 0 \end{aligned}$$

Also,

$$(6\text{-}23) \qquad \begin{aligned} {}_F\Omega_{22} &= \xi_1 \Gamma^1_{22} = \xi^1 \Gamma_{22,1} \\ {}_F\Omega_{33} &= \xi_1 \Gamma^1_{33} = \xi^1 \Gamma_{33,1} \\ {}_F\Omega_{23} &= \xi_1 \Gamma^1_{23} = \xi^1 \Gamma_{23,1} = 0 \end{aligned}$$

From (6-9),

$$(6\text{-}24) \qquad \begin{aligned} \Gamma_{22,1} &= -\frac{1}{2} \frac{\partial g_{22}}{\partial \chi} = a^2 \sin\chi \cos\chi \\ \Gamma_{33,1} &= -\frac{1}{2} \frac{\partial g_{33}}{\partial \chi} = a^2 \sin\chi \cos\chi \sin^2\theta \end{aligned}$$

Consequently, using (6-23) and (6-16),

$$(6\text{-}25) \qquad {}_F\Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a\sin\chi_0\cos\chi_0 & 0 \\ 0 & 0 & a\sin\chi_0\cos\chi_0\sin^2\theta \end{pmatrix}$$

Next we turn to the evaluation of ${}_K\Omega_{ij}$. From (6-16) and (6-18) we have:

$$(6\text{-}26) \qquad \begin{aligned} {}_K\Omega_{ij} = &- \frac{\partial \xi_0}{\partial x^\nu} \frac{\partial x^0}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^j} - \frac{\partial \xi_1}{\partial x^\nu} \frac{\partial x^1}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^j} \\ &+ \left(\Gamma^0_{\mu\nu}\xi_0 + \Gamma^1_{\mu\nu}\xi_1 \right) \frac{\partial x^\mu}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^j} \end{aligned}$$

Then

$$(6\text{-}27) \qquad \begin{aligned} {}_K\Omega_{12} = &- \frac{\partial \xi_0}{\partial \theta} \frac{dt}{d\omega} - \frac{\partial \xi_1}{\partial \theta} \frac{dr}{d\omega} + \Gamma^0_{02}\xi_0 \frac{dt}{d\omega} + \Gamma^1_{02}\xi_1 \frac{dt}{d\omega} \\ &+ \Gamma^0_{12}\xi_0 \frac{dr}{d\omega} + \Gamma^1_{12}\xi_1 \frac{dr}{d\omega} \end{aligned}$$

From (6-16),

(6-28) $$\frac{\partial \xi_0}{\partial \theta} = \frac{\partial \xi_1}{\partial \theta} = 0$$

From (6-9),

(6-29) $$\Gamma^{1}_{02}\,\xi_1 = \xi^1\,\Gamma_{02,1} = 0$$
$$\Gamma^{0}_{12}\,\xi_0 = \xi^0\,\Gamma_{12,0} = 0$$

Therefore, (6-27) becomes

(6-30) $$\kappa\,\Omega_{12} = \xi^0\,\Gamma_{02,0}\,\frac{dt}{d\omega} + \xi^1\,\Gamma_{12,1}\,\frac{dr}{d\omega}$$

But, again from (6-9),

(6-31) $$\Gamma_{02,0} = \frac{1}{2}\,\frac{\partial g_{00}}{\partial x^2} = 0$$
$$\Gamma_{12,1} = \frac{1}{2}\,\frac{\partial g_{11}}{\partial x^2} = 0$$

and finally

(6-32) $$\kappa\,\Omega_{12} = 0$$

Similarly, it follows that

(6-33) $$\kappa\,\Omega_{13} = 0$$

Returning to (6-26), we have

(6-34) $$\kappa\,\Omega_{22} = \Gamma^{0}_{22}\,\xi_0 + \Gamma^{1}_{22}\,\xi_1$$

(6-35) $$\kappa\,\Omega_{33} = \Gamma^{0}_{33}\,\xi_0 + \Gamma^{1}_{33}\,\xi_1$$

(6-36) $$\kappa\,\Omega_{23} = \Gamma^{0}_{23}\,\xi_0 + \Gamma^{1}_{23}\,\xi_1$$

since by (6-5)

(6-37) $$\frac{\partial x^0}{\partial \bar{x}^2} = \frac{\partial x^0}{\partial \bar{x}^3} = \frac{\partial x^1}{\partial \bar{x}^2} = \frac{\partial x^1}{\partial \bar{x}^3} = 0$$

From (6-9), we have

(6-38) $$\xi_0\,\Gamma^{0}_{23} + \Gamma^{1}_{23}\,\xi_1 = \xi^0\,\Gamma_{23,0} + \xi^1\,\Gamma_{23,1} = 0$$

so (6-36) becomes

(6-39) $$\kappa\,\Omega_{23} = 0$$

Equations (6-34) and (6-35) may be written as

(6-40) $$\kappa\,\Omega_{22} = \xi^0\,\Gamma_{22,0} + \xi^1\,\Gamma_{22,1}$$

(6-41) $$\kappa\,\Omega_{33} = \xi^0\,\Gamma_{33,0} + \xi^1\,\Gamma_{33,1}$$

Then, from (6-9),

$$(6\text{-}42) \qquad \Gamma_{22,0} = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^0} = 0$$

$$(6\text{-}43) \qquad \Gamma_{33,0} = -\frac{1}{2}\frac{\partial g_{33}}{\partial x^0} = 0$$

$$(6\text{-}44) \qquad \Gamma_{22,1} = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = r$$

$$(6\text{-}45) \qquad \Gamma_{33,1} = -\frac{1}{2}\frac{\partial g_{33}}{\partial x^1} = r\sin^2\theta$$

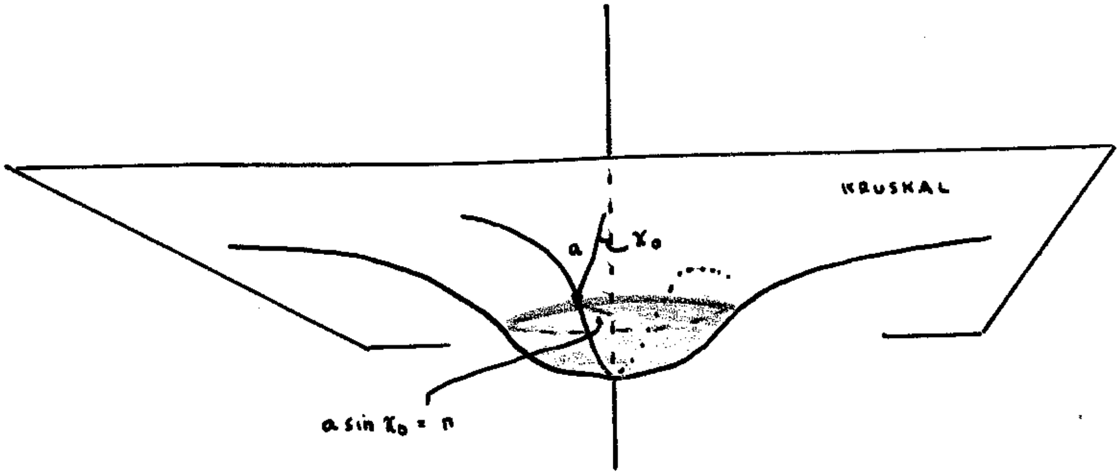
Using (6-16) we finally obtain

$$(6\text{-}46) \qquad {}_K\Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r\alpha & 0 \\ 0 & 0 & r\alpha\sin^2\theta \end{pmatrix}$$

Comparing (6-25) and (6-46) by means of the conditions (6-14) and (6-15), it is clear that ${}_F\Omega_{ij}$ and ${}_K\Omega_{ij}$ are identical if the positive sign is taken in (6-14). This means that $\chi_0 < \frac{\pi}{2}$. The matching is shown in figure 6-1. Now suppose we assume $\chi_0 > \frac{\pi}{2}$, so that we must use the negative sign in (6-14). If we replace ${}_K\xi$ in (6-16) by $-{}_K\xi$ we will change the signs of each component of as is evident from (6-40) and (6-41). Once again ${}_F\Omega_{ij}$ and ${}_K\Omega_{ij}$ are identical. But we must show that it is physically meaningful to align ${}_F\xi$ which points outward from the Friedmann universe with a vector field $-{}_K\xi$ which points into the Kruskal universe and yet also points in a direction of decreasing r. This is done in figure 6-2.

We now turn to the problem of matching a hyperbolic Friedmann universe to a Kruskal universe in which radial geodesics are characterized by an energy parameter α greater than unity. The calculations for this case will be streamlined considerably since the procedure is practically identical to that carried out above. The parametric equations (2-9) and (3-13) for radial geodesics are listed below for convenience:

FIGURE 6-1



The upper figure is a representation of the matching at some instant $v = v_0$. The θ - coordinate has been suppressed. The lower figure indicates that part of Kruskal spacetime which is joined to the spherical Friedmann universe. Note that in this diagram $\chi_0 < \frac{\pi}{2}$.

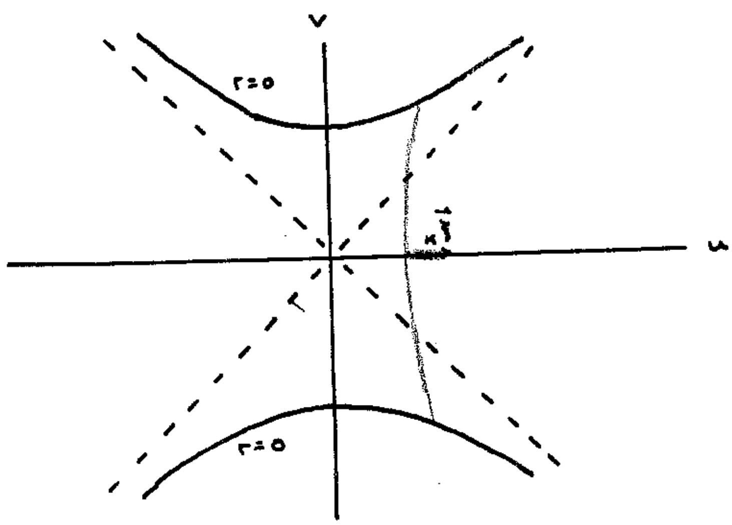
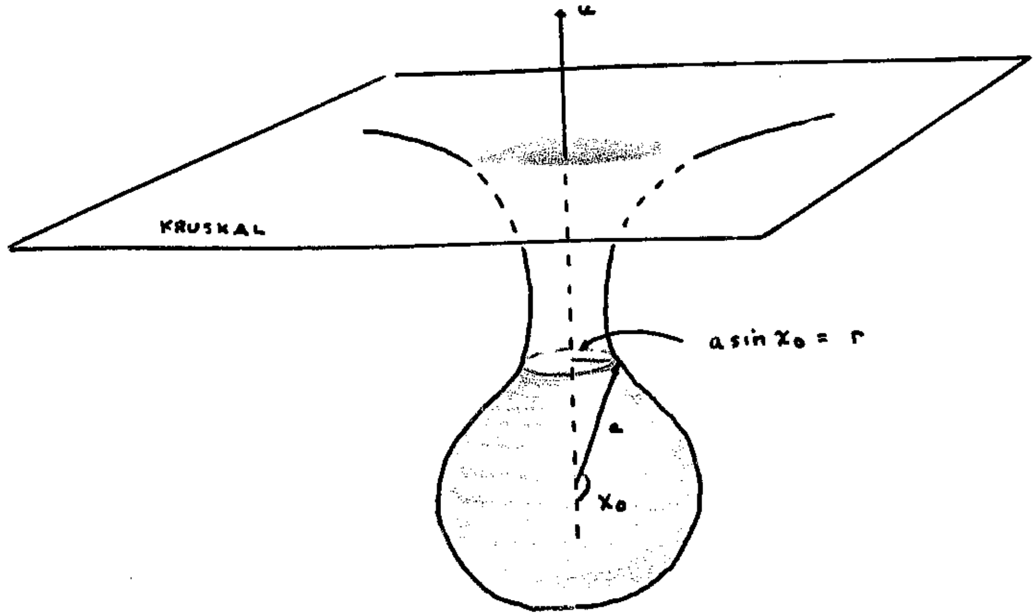
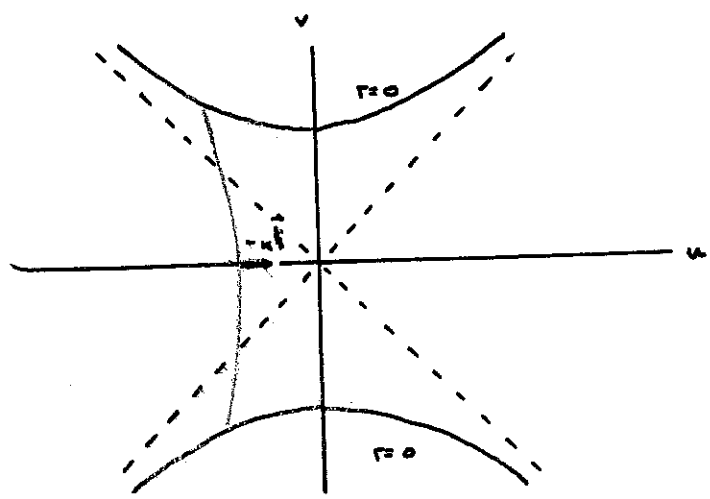


FIGURE 6-2



The upper figure is a representation of the matching at some instant $v = v_0$. The θ - coordinate has been suppressed. The lower figure indicates that part of Kruskal spacetime which is joined to the spherical Friedmann universe. Note that in this diagram $\chi_0 > \frac{\pi}{2}$.



$$(6\text{-}47)\quad \begin{aligned} {}_F U:\quad & a = a_0(\cosh \omega_F - 1) \\ & \tau_F = t = a_0(\sinh \omega_F - \omega_F) \\ {}_K U:\quad & r = \frac{M\kappa}{\alpha^2 - 1}(\cosh \omega_K - 1) \\ & \tau_K = \frac{M\kappa}{(\alpha^2-1)^{3/2}}(\sinh \omega_K - \omega_K) \end{aligned}$$

In order to identify τ_F and τ_K we must require that

$$(6\text{-}48)\quad \omega_F = \omega_K = \omega$$

and

$$(6\text{-}49)\quad a_0 = \frac{M\kappa}{(\alpha^2 - 1)^{3/2}}$$

Equation (6-49) will be discussed along with (6-3) in the next chapter. The equations for the boundary manifolds ${}_F B$ and ${}_K B$ are:

$$(6\text{-}50)\quad {}_F B:\quad \begin{aligned} t &= a_0(\sinh \omega - \omega) \\ \chi &= \chi_0 \\ \theta &= \theta \\ \varphi &= \varphi \end{aligned}$$

$$(6\text{-}51)\quad {}_K B:\quad \begin{aligned} t &= t(\omega) \\ r &= \frac{M\kappa}{\alpha^2 - 1}(\cosh \omega - 1) \\ \theta &= \theta \\ \varphi &= \varphi \end{aligned}$$

In (6-51), $t = t(\omega)$ must be determined from (3-9) and (3-13) which were

$$(6\text{-}52)\quad \begin{aligned} \tau &= \frac{M\kappa}{(\alpha^2-1)^{3/2}}(\sinh \omega - \omega) \\ \dot{t} &= \frac{dt}{d\tau} = \frac{\alpha}{X} \end{aligned}$$

Using the same notational conventions as before, we can easily evaluate the first fundamental forms for ${}_F B$ and ${}_K B$:

$$(6\text{-}53)\quad \begin{aligned} {}_F I &= a^2 d\omega^2 - a^2 \sinh^2 \chi_0 (d\theta^2 + \sin^2\theta\, d\varphi^2) \\ {}_K I &= \frac{r^2}{\alpha^2 - 1} d\omega^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \end{aligned}$$

After identifying the coordinates ω , θ , and φ we must have:

(6-54) $$r^2 = a^2 \sinh^2 \chi_0$$

(6-55) $$\frac{r^2}{\alpha^2 - 1} = a^2$$

It follows that

(6-56) $$\sinh^2 \chi_0 = \alpha^2 - 1$$

or

(6-57) $$\cosh \chi_0 = + \alpha$$

where the positive sign must be taken because cosh is a positive function. This gives one condition from the two equations (6-54) and (6-55). We take the other condition to be

(6-58) $$r = a \sinh \chi_0$$

which is automatically satisfied if (6-48), (6-49), and (6-56) are.

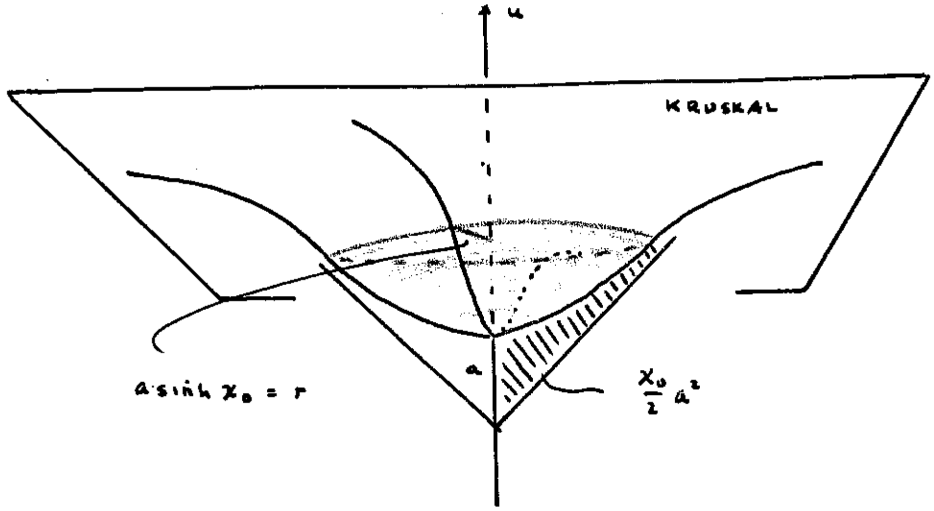
The calculation of the second fundamental forms of $_F B$ and $_K B$ proceeds exactly as before except that sin χ and cos χ are to be replaced everywhere by sinh χ and cosh χ : The final results are:

(6-59) $$_F\Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a \sinh \chi_0 \cosh \chi_0 & 0 \\ 0 & 0 & a \sinh \chi_0 \cosh \chi_0 \sin^2\theta \end{pmatrix}$$

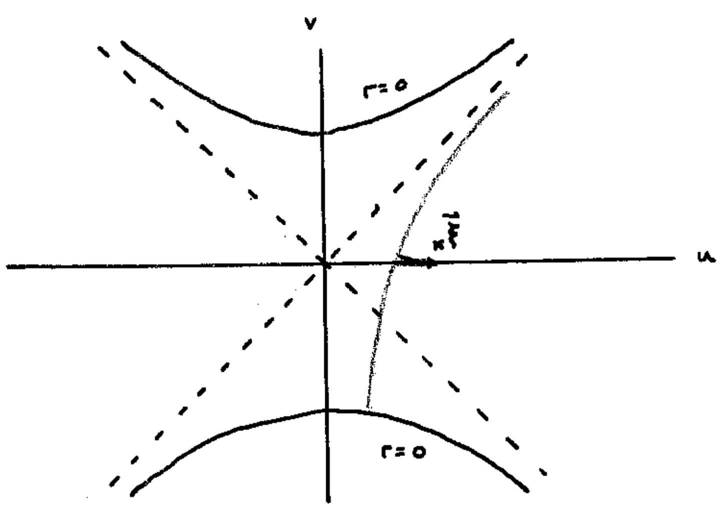
(6-60) $$_K\Omega_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r\alpha & 0 \\ 0 & 0 & r\alpha \sin^2\theta \end{pmatrix}$$

It is apparent that (6-59) and (6-60) are identical if the conditions (6-57) and (6-58) are satisfied. The matching is shown in figure 6-3.

FIGURE 6-3



The upper figure is a representation of the matching at some instant $v = v_0$. The θ - coordinate has been suppressed. The lower figure indicates that part of Kruskal spacetime which is joined to the hyperbolic Friedmann universe.



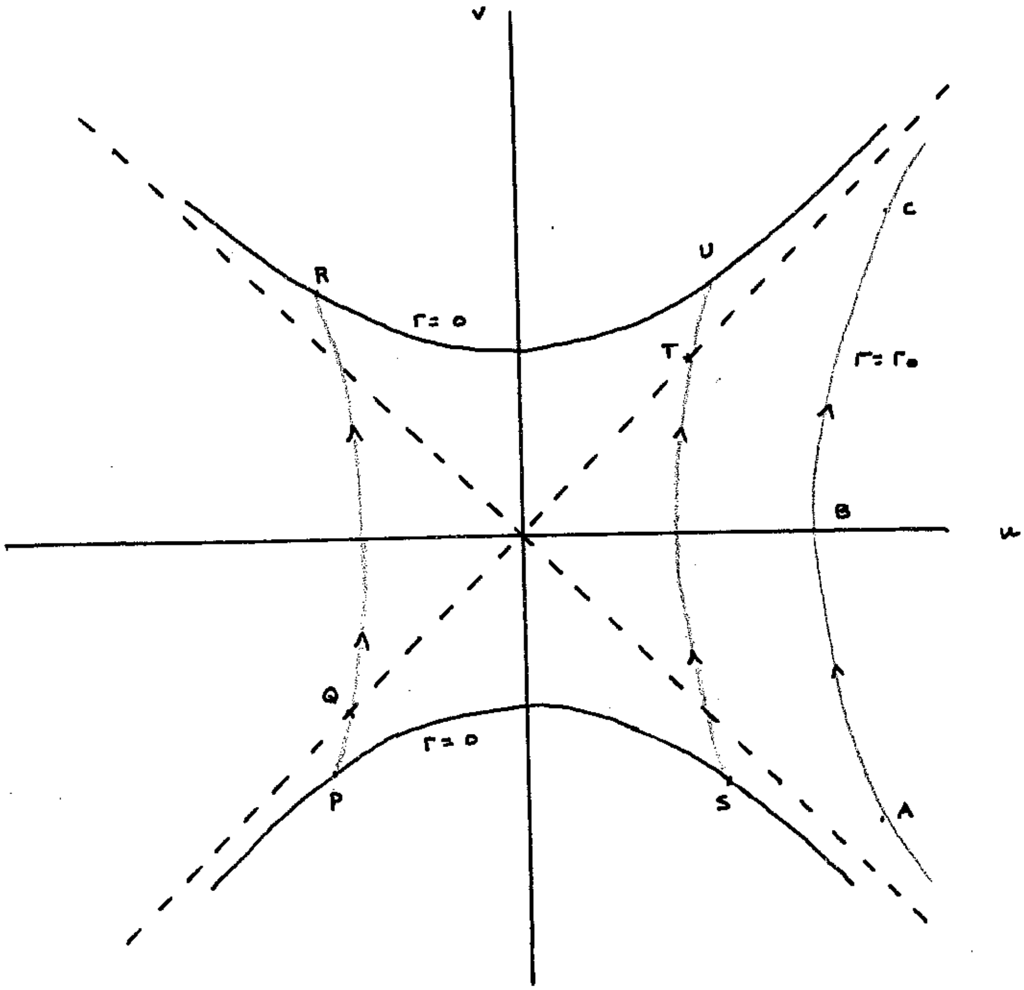
CONCLUSIONS

In figure 7-1 we have combined the matchings illustrated in figures 6-1 and 6-2 into a single Kruskal coordinate diagram. The boundary manifold B of the universe U obtained by identifying $_F U$ and $_K U$ along $_F B$ and $_K B$ travels along one of the radial geodesics PQR or STU. The path of an observer who maintains himself at a constant "distance" r_0 from the center of the mass distribution is also shown. This observer could be a person on a planet revolving about the central "star" or Friedmann universe. Recalling that null geodesics are lines with slope ± 1 , we see that as the exterior observer passes in the indicated direction along ABC his proper time changes, increasing toward plus infinity, and he sees the boundary B of the central "star" asymptotically approach either Q or T depending upon whether B travels along PQR or STU, that is to say, depending upon the value of χ_0 chosen for the matching. This means that the observer sees r of B approach the limit $2M_K$ asymptotically. Also, since a smaller and smaller element of path for B corresponds to a greater and greater element of path for the observer as B nears either Q or T, the visual magnitude of the star decreases to zero as the observer's proper time increases without bound. It is to be noted that, in contrast to the perceptions of our exterior observer, an observer located on the surface of the star itself would see the star expand and then contract, becoming singular after a finite proper time[1].

Now let us consider equation (6-3), which was,

$$(7\text{-}1) \qquad a_0 = \frac{M_K}{(1-\alpha^2)^{3/2}}$$

FIGURE 7-1



PQR and STU are radial geodesics along which the boundary $B \equiv {}_F B \equiv {}_K B$ travels. PQR corresponds to $\chi_0 > \frac{\pi}{2}$; STU, to $\chi_0 < \frac{\pi}{2}$. An exterior observer, maintaining himself at constant r , travels along the curve ABC and, as indicated, sees only the arcs PQ of PQR and ST of STU.

From (2-20) we have

(7-2) $$a_0 = \frac{4M_F}{3(2\chi_0 - \sin 2\chi_0)}$$

Consequently,

(7-3) $$M_K = \frac{4(1-\alpha^2)^{3/2}}{3(2\chi_0 - \sin 2\chi_0)} M_F$$

Applying the matching condition (6-13)

(7-4) $$M_K = \frac{4\sin^3\chi_0}{3(2\chi_0 - \sin 2\chi_0)} M_F$$

$\frac{M_K}{M_F}$ is plotted as a function of χ_0 in figure 7-2. For small angles,

(7-5) $$\chi_0 \ll 1 ,$$

we have for a first order approximation to (7-4),

(7-6) $$M_K = \frac{4\chi_0^3}{3\left[2\chi_0 - 2\chi_0 + \frac{(2\chi_0)^3}{3!}\right]} M_F$$

which reduces to

(7-7) $$M_K = M_F$$

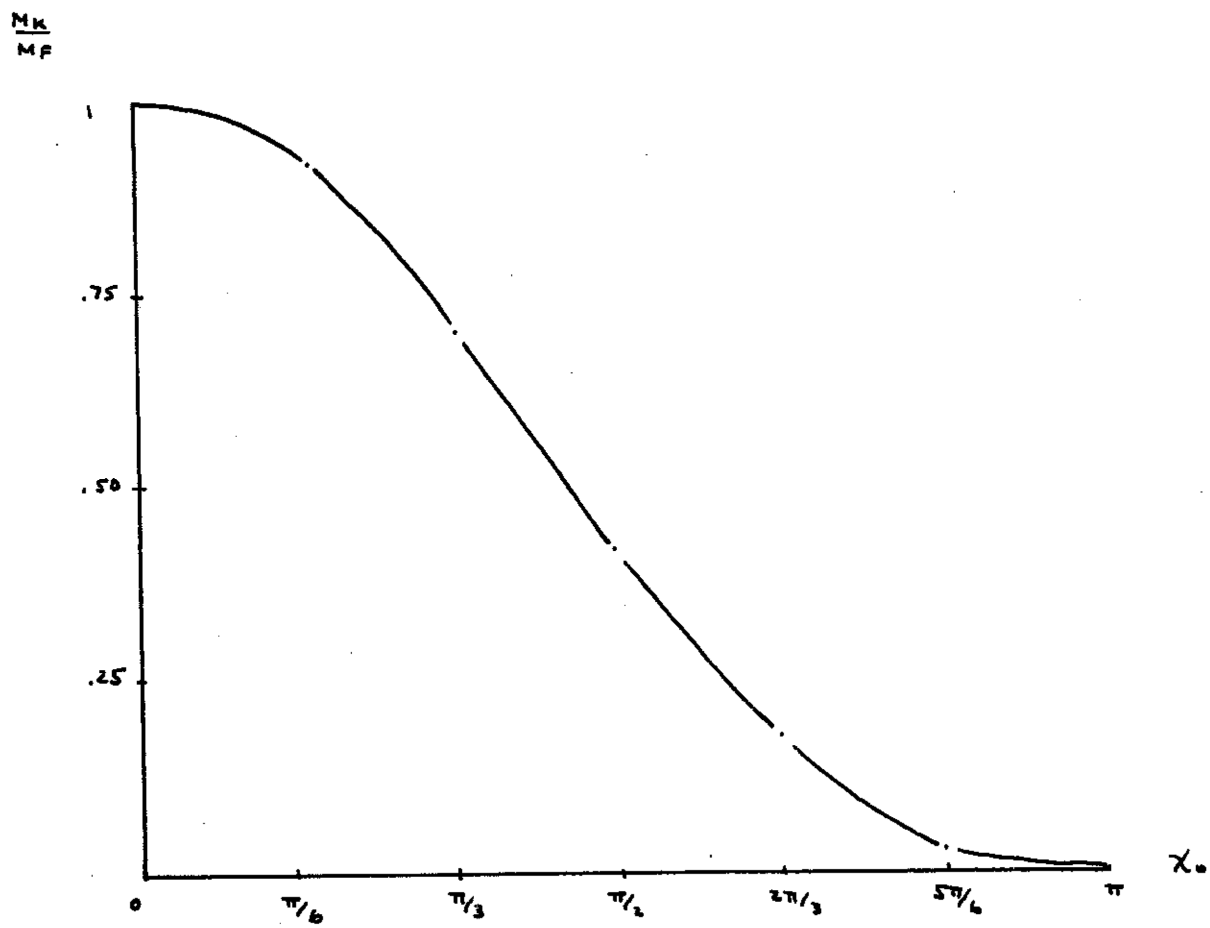
This seems quite reasonable since, in this approximation, we have neglected the kinetic and potential energies of particles in the Friedmann universe as seen by an observer in the Kruskal universe. If we now pass on to a second order approximation, (7-4) becomes

(7-8) $$M_K = \frac{4\left[\chi_0^3 - \frac{\chi_0^5}{2}\right]}{3\left[2\chi_0 - 2\chi_0 + \frac{(2\chi_0)^3}{3!} - \frac{(2\chi_0)^5}{5!}\right]} M_F$$

which reduces to

(7-9) $$M_K = \frac{1 - \frac{\chi_0^2}{2}}{1 - \frac{\chi_0^2}{5}} M_F$$

FIGURE 7-2



The ratio $\frac{M_K}{M_F}$ is plotted against χ_o
for spherical Friedmann universes

Expanding the denominator and multiplying, we obtain

(7-10) $$M_K = \left[1 - \frac{3}{10}\chi_0^2\right] M_F$$

But

(7-11) $$\chi_0^2 \doteq \sin^2\chi_0 \doteq 1-\alpha^2$$

To evaluate α, we recall (3-11)

(7-12) $$\sqrt{\alpha^2 - X} = \frac{dr}{d\tau}$$

and since, at the moment of maximum expansion,

(7-13) $$\frac{dr}{d\tau} = 0$$

we have

(7-14) $$\alpha^2 = 1 - \frac{2M_K}{r_{max}}$$

Using (7-11) and (7-14) in (7-10), we find that

(7-15) $$M_K = \left[1 - \frac{3}{5}\frac{M_K}{r_{max}}\right] M_F$$

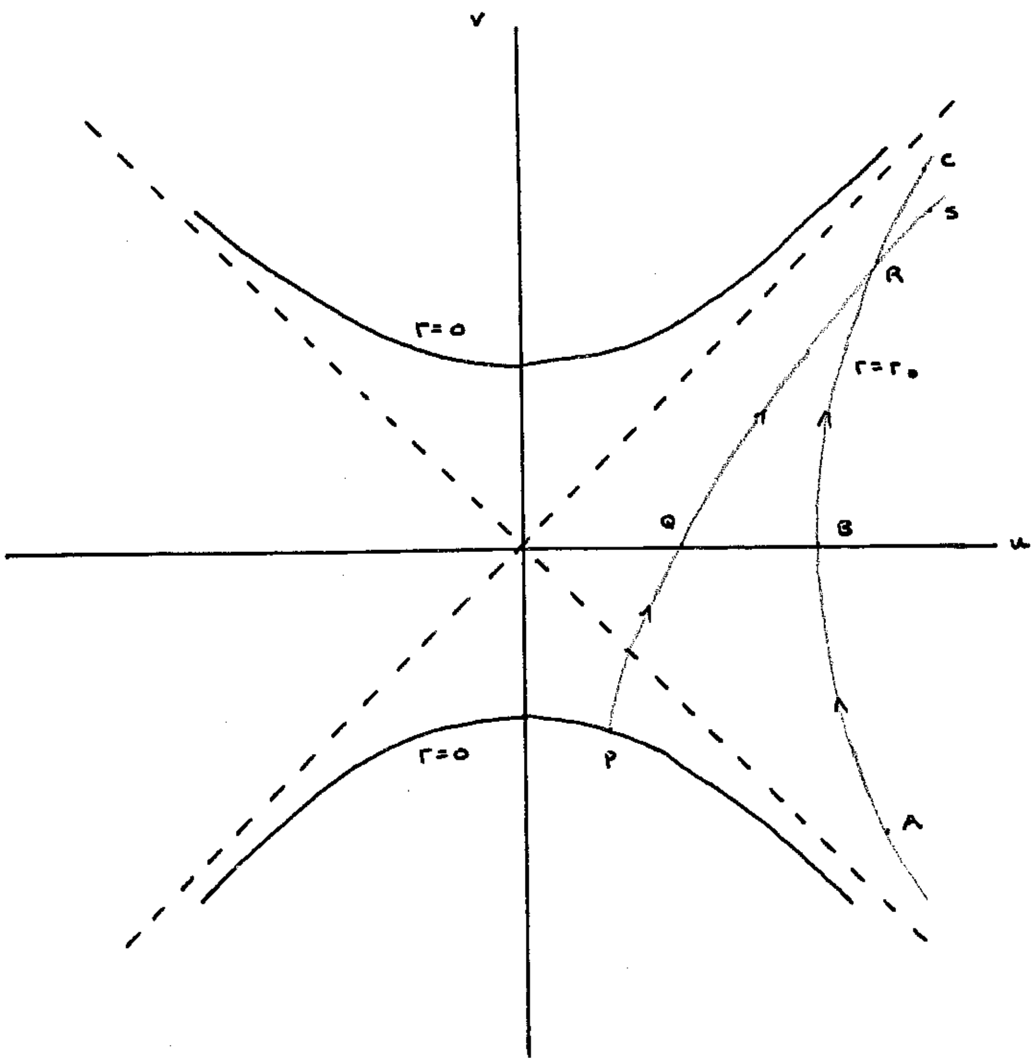
Because of (7-7), at this level of approximation, (7-15) becomes

(7-16) $$M_K = M_F - \frac{3}{5}\frac{M_F^2}{r_{max}}$$

The second term on the right corresponds to the potential energy of the dust particles composing the spherical Friedmann universe, as would be calculated by Newtonian mechanics,[24] so we may conclude that our relativistic star model reduces to a Newtonian model if (7-5) is satisfied.

Now let us turn to the case of a hyperbolic Friedmann universe. In figure 7-3 the radial geodesic followed by the boundary B has been indicated. In this case it is clear that the exterior observer sees the star expand without limit, eventually enveloping the observer himself.

FIGURE 7-3



PQRS is the radial geodesic followed by the boundary B. ABRC is the path followed by the exterior observer at some constant r. The observer is enveloped by the expanding star at R.

Finally, let us examine equation (6-49) briefly:

$$(7\text{-}17) \qquad a_0 = \frac{M_K}{(\alpha^2-1)^{3/2}}$$

From (2-21)

$$(7\text{-}18) \qquad a_0 = \frac{4M_F}{3(\sinh 2\chi_0 - 2\chi_0)}$$

Therefore, using (6-56)

$$(7\text{-}19) \qquad M_K = \frac{4\sinh^3 \chi_0}{3(\sinh 2\chi_0 - 2\chi_0)} M_F$$

The small angle approximations considered for the spherical case may be carried out in this case as well. The first order approximation clearly leads to the same result, that the masses M_F and M_K are equal. If we recall that

$$(7\text{-}20) \qquad \sqrt{\alpha^2 - \chi} = \frac{dr}{d\tau}$$

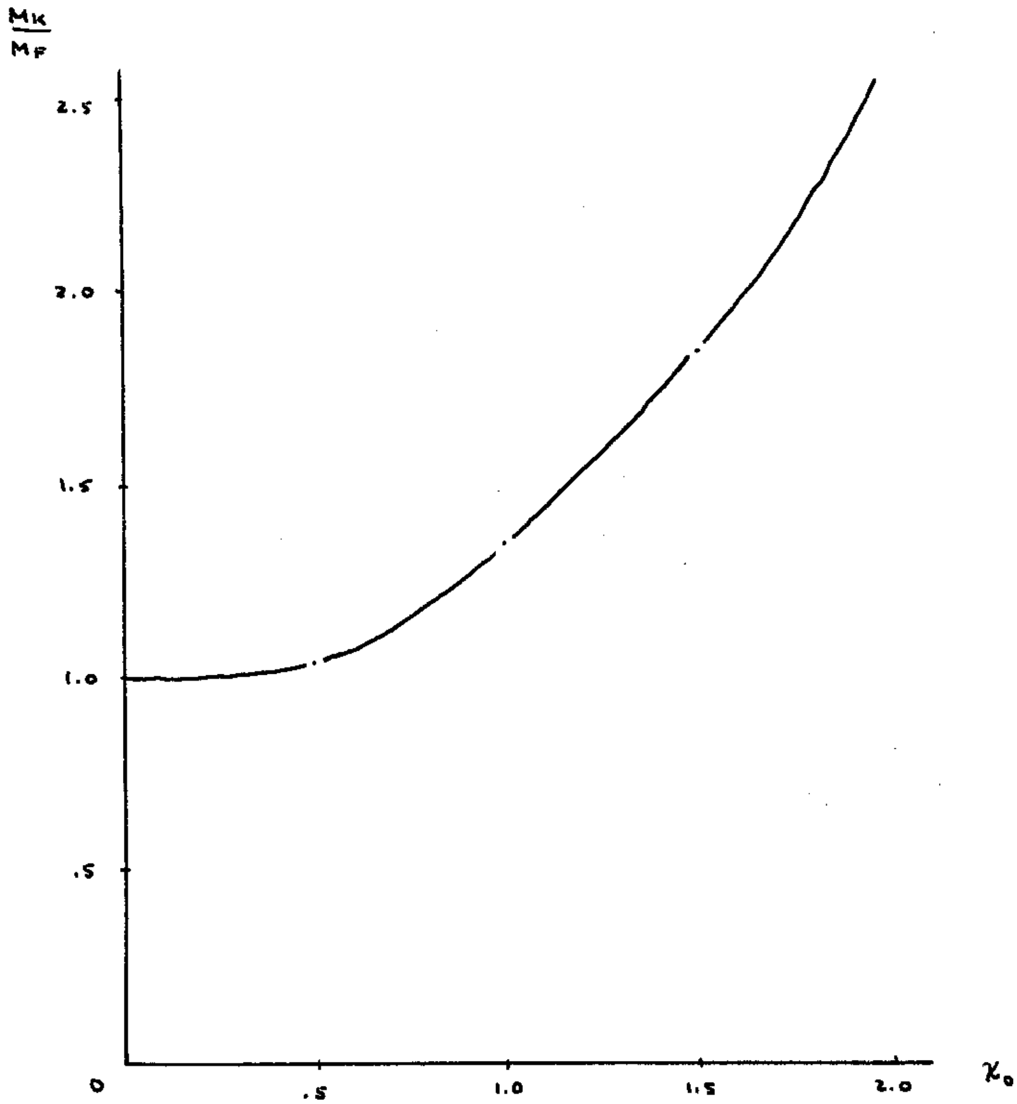
the second order approximation leads to the conclusion that

$$(7\text{-}21) \qquad M_K = M_F - \frac{3}{5}\frac{M_F^2}{r} + \frac{3}{10}\left(\frac{dr}{d\tau}\right)^2 M_F$$

where the second term on the right is to be interpreted as a potential energy contribution and the third term on the right, as a kinetic energy contribution. The ratio $\frac{M_K}{M_F}$ for hyperbolic Friedmann universes is plotted in figure 7-4.

David L. Beckedorff

FIGURE 7-4



The ratio $\frac{M_K}{M_F}$ is plotted against χ_0 for hyperbolic Friedmann universes

NOTES

1. [25], pp. 187 - 188.
2. [19].
3. [13], pp. 138 - 140.
4. [25], p. 189.
5. [16].
6. [25], pp. 187 - 188.
7. [25], p. 184.
8. [11] .
9. [12], pp. 332 - 344.
10. [24], pp. 253 - 257.
11. [24], pp. 252 - 253.
12. [24], pp. 239 - 245.
13. [15], pp. 27 - 29.
14. [5], [7], [8], [11].
15. [7].
16. [8].
17. [11].
18. [5].
19. [25], p. 131.
20. [6], p. 143.
21. [6], p. 144.
22. [6], p. 155.
23. [6], p. 155.
24. [13], p. 127.

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