① • Let $p^\mu \equiv \frac{dx^\mu}{d\lambda}$ be the 4-momentum of a massless particle.

Geodesic eq.

$$p^\alpha \nabla_\alpha p_\mu = 0 = \underbrace{p^\alpha \partial_\alpha p_\mu}_{\frac{d}{d\lambda} p_\mu(\lambda)} - \underbrace{\Gamma^\nu_{\alpha\mu} p_\nu p^\alpha}_{\frac{1}{2} g^{\nu\beta}(\partial_\alpha g_{\beta\mu} + \partial_\mu g_{\beta\alpha} - \partial_\beta g_{\alpha\mu}) p_\nu p^\alpha =}$$

$$= \frac{1}{2}\left[\cancel{(\partial_\alpha g_{\beta\mu}) p^\alpha p^\beta} + (\partial_\mu g_{\beta\alpha}) p^\alpha p^\beta - \cancel{(\partial_\alpha g_{\beta\mu}) p^\alpha p^\beta}\right] =$$

$$= \frac{1}{2} p^\alpha p^\beta \partial_\mu g_{\alpha\beta}$$

$$\Rightarrow \frac{dp_\mu}{d\lambda} = \frac{1}{2} p^\alpha p^\beta \partial_\mu g_{\alpha\beta}$$

Let $\mu = 0$: $\frac{dp_0}{d\lambda} = \frac{1}{2} p^\alpha p^\beta \underbrace{\partial_0 g_{\alpha\beta}}_{\text{0 by assumption}} = 0 \Rightarrow p_0$ conserved along geodesic

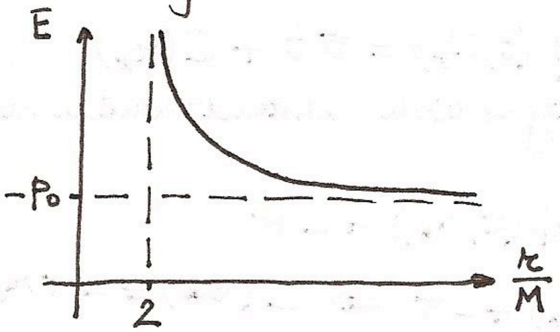
• Observer at rest: $u^\mu_{obs} = (u^t, 0, 0, 0)$

$$u^\mu_{obs} u_\mu^{obs} = -1 \Rightarrow -\left(1 - \frac{2M}{r}\right)(u^t)^2 = -1 \Rightarrow u^t = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$$

Let the graviton have 4-momentum p^μ. The observer measures (see 5.100 in Carrol):

$$E = -g_{\mu\nu} u^\mu_{obs} p^\nu = -g_{tt} u^t p^t = -u^t p_0 = -\frac{p_0}{\sqrt{1 - \frac{2M}{r}}} \quad \text{when } p_0 = \text{const.}$$

As the graviton travels to larger r, the measured E is lower and lower, i.e. more and more redshifted



• Geometrical optics. Radial rays ($d\Omega = 0$). In tortoise coords. the metric reads (see 5.109 in Carroll):

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr_*^2\right) \qquad (\#I)$$

Let

$$\bar{h}_{\mu\nu} = A(t, r_*)\, e^{i\phi(t, r_*)}\, \epsilon_{\mu\nu}$$

when $\epsilon_{\mu\nu}$ is the polarization tensor, s.t. $\epsilon^*_{\mu\nu}\epsilon^{\mu\nu} = 1$. In the eikonal approximation

$$g_{\mu\nu} K^\mu K^\nu = 0 \qquad \text{(Maggiore 1.187)}$$

where $K_\mu \equiv \partial_\mu \phi(t, r_*)$. Then

$$g^{tt}(\partial_t \phi)^2 + g^{r_* r_*}(\partial_{r_*}\phi)^2 = -\frac{1}{1-\frac{2M}{r}}(\partial_t\phi)^2 + \frac{1}{1-\frac{2M}{r}}(\partial_{r_*}\phi)^2 = 0$$

$$\Rightarrow \partial_t \phi(t, r_*) = \pm\, \partial_{r_*}\phi(t, r_*) \qquad (\#II)$$

If $\phi = \phi(t - r_*)$ then eq. (#II) is satisfied since

$$\partial_t \phi(t - r_*) = \partial_{t-r_*}\phi(t - r_*) =$$

$$= -\partial_{r_*-t}\phi(t - r_*) =$$

$$= -\partial_{r_*}\phi(t - r_*)$$

Also K_μ must obey the geodesic eq. (see 1.188 in Maggiore)

$$K^\alpha \bar{D}_\alpha K_\mu = 0$$

Since the metric (#I) does not depend on time $(\partial_0 g_{\mu\nu} = 0)$ from the results of the 1st part we have:

$$K_0 = \text{const.}$$

This means that

$$K_0 = \partial_t \phi = \text{const.} \equiv \sigma$$

$$\Rightarrow \phi(t, r_*) = \sigma t + C(r_*)$$

with $\sigma = $ const. and C an integration constant which depend only on r_*. But from (#II)

$$\partial_{r_*}\phi(t, r_*) = -\partial_t \phi(t, r_*) = -\sigma$$

$$\Rightarrow C'(r_*) = -\sigma \Rightarrow C(r_*) = -\sigma r_* + \delta$$ where δ is a pure constant of integration

$$\Rightarrow \phi(t, r_*) = \sigma(t - r_*) + \delta \quad \checkmark$$

• Then from the 2^{nd} part of the problem, noting that $\hbar k_\mu$ is the 4-momentum of the propagating graviton:

$$E = -\frac{\hbar K_0}{\sqrt{1-\frac{2M}{r}}} = -\frac{\hbar \partial_t \phi}{\sqrt{1-\frac{2M}{r}}} = -\frac{\hbar \sigma}{\sqrt{1-\frac{2M}{r}}} \quad \text{(in geometric units } G=c=1\text{)}$$

The frequency ω of the graviton is related to the energy E:

$$E = \hbar \omega \Rightarrow \omega = -\frac{\sigma}{\sqrt{1-2M/r}}$$

• The waveform has amplitude $\propto 1/r$ and phase $\phi = \sigma(t-r_*)+\delta$

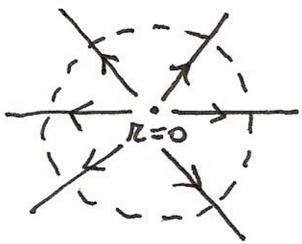
$$h = \mathrm{Re}\left\{\frac{\tilde{A}}{r}\exp[i\phi]\right\} \quad \text{where } \tilde{A} \text{ is a const.}$$

$$= \frac{\tilde{A}}{r}\cos[\sigma(t-r_*)+\delta]$$

To show that the scalar amplitude A goes like $1/r$, we can use the conservation of graviton flux of geometric optic, which can be written as

$$\mathcal{A} A^2 = \text{const}$$

where $\mathcal{A}$ is the cross sectional area of a bundle of rays. See MTW exercise 22.13. Here we take rays moving radially in all directions, away from $r=0$. Then this bundle of rays has a spherical surface as cross section:



$$\mathcal{A} = 4\pi r^2$$

$$\Rightarrow 4\pi r^2 A^2 = \text{const.}$$

$$\Rightarrow A \propto \frac{1}{r}$$

② $(1.4 + 1.4) M_\odot$ on circular orbit.

• $P = 7.75\,h$. Derive eq. (4) first.

Orbital energy is $E = -\frac{G m_1 m_2}{2R}$ where R is the orbital radius.

Let $\omega = \frac{2\pi}{P}$ be the orbital freq of the binary.

Kepler's law is $\omega^2 = \frac{G(m_1+m_2)}{R^3}$. For a quasi-circular inspiral we can assume that Kepler's law holds throughout the inspiral.

$$\dot R = -\frac{2}{3}\frac{G(m_1+m_2)}{\omega^{2/3}}\frac{\dot\omega}{\omega} = -\frac{2}{3}\frac{R\dot\omega}{\omega}, \qquad \dot\omega = -\frac{2\pi}{P^2}\dot P$$

$$\dot E = \frac{G m_1 m_2}{2R^2}\dot R = -\frac{G m_1 m_2}{3R}\frac{\dot\omega}{\omega} = -\frac{G m_1 m_2}{3}\frac{\omega^{2/3}}{G^{1/3}(m_1+m_2)^{1/3}}\frac{\dot\omega}{\omega} =$$

$$= -\frac{G^{2/3}}{3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\frac{\dot\omega}{\omega^{1/3}} = -\frac{G^{2/3}}{3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\left(-\frac{2\pi}{P}\frac{\dot P}{P}\right)\frac{P^{1/3}}{(2\pi)^{1/3}} =$$

$$= +\frac{(2\pi)^{2/3}G^{2/3}}{3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\frac{\dot P}{P^{5/3}}$$

$$\underbrace{\dot E = -P_{GW}}_{\text{balance eq.}} = -\frac{32}{5}\frac{c^5}{G}\frac{G^{10/3}}{\cancel{2^{10/3}}}\frac{M_c^{10/3}}{c^{10}}(\cancel{2}\omega)^{10/3}, \quad \text{where } M_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

quadrupole formula (Maggiore 4.12)

$$= -\frac{32}{5}\frac{G^{7/3}}{c^5}\frac{(m_1 m_2)^2}{(m_1+m_2)^{2/3}}\frac{(2\pi)^{10/3}}{P^{10/3}}$$

Then equating:

$$\frac{\cancel{(2\pi)^{2/3}}\,\cancel{G^{2/3}}}{3}\frac{\cancel{(m_1 m_2)}}{\cancel{(m_1+m_2)^{1/3}}}\dot P\Big/\cancel{P^{5/3}} = -\frac{32}{5}\frac{G^{\cancel{7/3}\,5/3}}{c^5}\frac{(m_1 m_2)^{\cancel{2}}}{(m_1+m_2)^{\cancel{2/3}\,1/3}}\frac{(2\pi)^{\cancel{10/3}\,8/3}}{P^{\cancel{10/3}\,5/3}}$$

$$\Rightarrow \boxed{\dot P = -\frac{96}{5}\frac{G^{5/3}}{c^5}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\frac{(2\pi)^{8/3}}{P^{5/3}}} \quad (\#\#)$$

Eq. (4) in the problem is

$$\dot P = -\frac{192\pi}{5}\frac{m_1 m_2}{(m_1+m_2)^2}\left(\frac{2\pi G(m_1+m_2)}{c^3 P}\right)^{5/3} = -\frac{\overset{96(2\pi)}{192\pi}}{5}\frac{m_1 m_2}{(m_1+m_2)^2}\frac{(2\pi)^{5/3}G^{5/3}(m_1+m_2)^{5/3}}{c^5 P^{5/3}}$$

$$= -\frac{96}{5}\frac{G^{5/3}}{c^5}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\frac{(2\pi)^{8/3}}{P^{5/3}} \quad \text{in agreement with } (\#\#)$$

Let's put $\dot P = -A P^{-5/3}$, where A collects all the factors in front of $P^{-5/3}$. Let $\tau \equiv t_c - t$, where t_c is the time of coalescence; here $P(t_c) = 0$

$$P^{5/3}\,dP = -A\,dt = A\,d\tau$$

$$\frac{P^{8/3}(\tau) - P^{8/3}(0)}{8/3} = A\tau$$

$$\Rightarrow \frac{3}{8A} P^{8/3}(\tau) = \tau$$

$$\boxed{\tau = \frac{3}{8}\,\frac{5}{96}\,\frac{c^5}{G^{5/3}}\,\frac{(m_1+m_2)^{1/3}}{m_1 m_2}\,\frac{1}{(2\pi)^{8/3}}\,P^{8/3}} = \quad (*)$$

$$= \frac{5}{256}\,\frac{c^5}{G^{5/3}}\,\frac{(m_1+m_2)^{1/3}}{m_1 m_2}\left(\frac{P}{2\pi}\right)^{8/3} =$$

$$= \frac{5}{256}\,\frac{(3\times10^8)^5}{(6.67\times10^{-11})^{5/3}}\,\frac{(2.8\,M_\odot)^{1/3}}{1.4^2\,M_\odot^2}\left(\frac{7.75\times60\times60}{2\pi}\right)^{8/3} s = \quad \text{use } M_\odot = 2\times10^{30}\,kg$$

$$= 5.2\times10^{16}\,s = 1.65\times10^9\,yrs = 1650\,Myr \quad \text{time to coalescence}$$

$$\dot{P} = -\frac{96}{5}\,\frac{(6.67\times10^{-11})^{5/3}}{(3\times10^8)^5}\,\frac{1.4^2\,M_\odot^2}{(2.8\,M_\odot)^{1/3}}\,\frac{(2\pi)^{8/3}}{(7.75\times60\times60)^{5/3}} = \quad \text{it's dimensionless}$$

$$= -2\times10^{-13}\,\frac{s}{s} = -2\times10^{-13}\,\frac{10^6\,\mu s}{\frac{yr}{365\times24\times3600}} =$$

$$= -6.3\,\frac{\mu s}{yr}$$

. Rewrite $(*)$ in terms of R.

$$\left(\frac{2\pi}{P}\right)^2 = \frac{G(m_1+m_2)}{R^3} \Rightarrow P = (2\pi)\,\frac{R^{3/2}}{G^{1/2}(m_1+m_2)^{1/2}}$$

$$\tau = \frac{5}{256}\,\frac{c^5}{G^{5/3}}\,\frac{(m_1+m_2)^{1/3}}{m_1 m_2}\,\frac{R^4}{G^{4/3}(m_1+m_2)^{4/3}} =$$

$$= \frac{5}{256}\,\frac{c^5}{G^3}\,\frac{R^4}{m_1 m_2 (m_1+m_2)} \quad \text{(consistent with 4.26 in Maggiore)}$$

$$\Rightarrow 2R = 2\left[\frac{256}{5}\,\frac{G^3}{c^5}\,m_1 m_2 (m_1+m_2)\,\tau\right]^{1/4} = \quad \text{let } \tau = 10^{10}\,yr$$

$$= 2\left[\frac{256}{5}\,\frac{(6.67\times10^{-11})^3}{(3\times10^8)^5}\,1.4^2 M_\odot^2\ 2.8 M_\odot\ 10^{10}\times365\times24\times3600\right]^{1/4} m =$$

$\cong 6.1\times10^9\ m \Rightarrow$ in order to have coalescence in less than $10^{10}\,yr$, the 2 stars must be closer then $\sim 6.1\times10^9\,m$

• Rewrite (*) in terms of f_{GW}:

$$\left.\begin{array}{l} \omega_{GW} = 2\omega \\ f_{GW} = \dfrac{\omega_{GW}}{2\pi} \end{array}\right\} \quad \omega = \frac{\omega_{GW}}{2} = \pi f_{GW}$$

$$\tau = \frac{5}{256} \frac{c^5}{G^{5/3}} \frac{(m_1+m_2)^{1/3}}{m_1 m_2} \omega^{-8/3} =$$

$$= \frac{5}{256} \frac{c^5}{G^{5/3}} \frac{(m_1+m_2)^{1/3}}{m_1 m_2} \frac{1}{(\pi f_{GW})^{8/3}} = 4.62 \times 10^5\,\mathrm{s} \left(\frac{1\,\mathrm{Hz}}{f_{GW}}\right)^{8/3}$$

$$\tau\Big|_{40\,\mathrm{Hz}} \cong 25\,\mathrm{s}$$

$$\tau\Big|_{100\,\mathrm{Hz}} \cong 2\,\mathrm{s}$$

• $r = 1\,\mathrm{Mpc} = 3.1 \times 10^{22}\,\mathrm{m}$

$$M_c = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \frac{1.4}{2^{1/5}} M_\odot$$

From Maggiore 4.20:

$$f_{GW}(\tau) = 134\,\mathrm{Hz} \left(\frac{1.21 M_\odot}{M_c}\right)^{5/8} \left(\frac{1s}{\tau}\right)^{3/8} =$$

$$\cong 134\,\mathrm{Hz} \left(\frac{1s}{t_c - t}\right)^{3/8}$$

The plot range is

$$\left.\begin{aligned} 100\,\mathrm{Hz} &\stackrel{!}{=} 134\,\mathrm{Hz} \left(\frac{1s}{\tau}\right)^{3/8} \Longrightarrow \tau = 2.18\,s \\ 300\,\mathrm{Hz} &\stackrel{!}{=} 134\,\mathrm{Hz} \left(\frac{1s}{\tau}\right)^{3/8} \Longrightarrow \tau = 0.12\,s \end{aligned}\right\} \quad 0.12\,s \leq \tau \leq 2.18\,s$$

From Maggiore 4.31, using $i = 0$:

$$h_+(\tau) = \frac{1}{r} \left(\frac{GM_c}{c^2}\right)^{5/4} \left(\frac{5}{c\tau}\right)^{1/4} \cos\left[-2\left(\frac{5GM_c}{c^3}\right)^{-5/8} \tau^{5/8} + \Phi_0\right] =$$

$$= 4.3 \times 10^{-21} \left(\frac{1s}{t_c - t}\right)^{1/4} \cos\left[1340\left(\frac{t_c - t}{1s}\right)^{5/8}\right]$$

In[544]:=
```
Plot[134 / (-t) ^ (3 / 8), {t, -0.12, -2.18},
  ImageSize → 300, AspectRatio → 1, PlotStyle → {Black, Thick},
  Frame → True, Axes → False, FrameLabel → {"t-t_c (sec)", "f_GW (Hz)"},
  GridLines → {{}, {0}}, LabelStyle → Medium, PlotRange → All]
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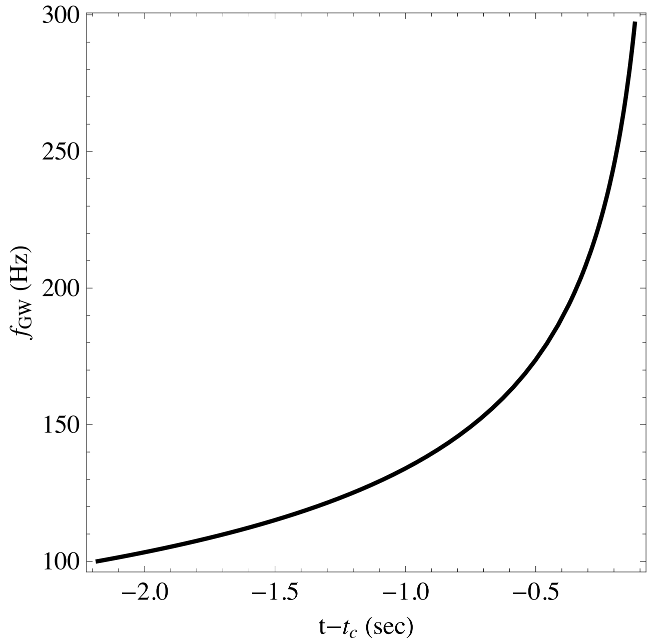
Out[544]=



In[543]:=
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Plot[1 / (-t) ^ (1 / 4) Cos[1340 (-t) ^ (5 / 8)], {t, -0.12, -2.18},
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  Axes → False, FrameLabel → {"t-t_c (sec)", "h_+/h_0"}, GridLines → {{}, {0}},
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Out[543]=