Homework: Weeks 3 & 4

Recommended readings:

1. C. Cutler et al., *The last three minutes*, Phys. Rev. Lett. **70**, 2894 (1993)
2. D. Kennefick, *Controversies in the history of the radiation reaction problem in general relativity*, gr-qc/9704002.

Assignment to be turned in at the beginning of the class on Tuesday, February 26 by students registered to the course:

- State what of the above readings you have done.
- Work the two exercises below.

Exercises:

1. Gravitational redshift of gravitational waves (10 points)
   - Let us consider a massless particle moving in a given spacetime metric $g_{\mu\nu}$. Assuming that $g_{\alpha\beta,0} = 0$ show that the component $p_0 = g_{0\mu} p^\mu$ of the particle's 4-momentum is conserved. [Hint: use the geodesic equation for the massless particle.]
   - Now, consider gravitational waves traveling through the spacetime of a non-spinning black hole. In appropriate coordinates (t, r, θ, ϕ) the spacetime metric has the Schwarzschild form

     $$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2\, d\Omega^2 \,. \tag{1}$$

     Let the gravitational waves have a reduced wavelength small compared to the hole's size, $\lambda \ll 2M$, and small compared to the radii of curvature of their phase fronts. Thus, geometric optics is a good approximation.

     Consider a graviton moving along a ray of the waves and a family of observers who are at rest with respect to the black hole. Compute the 4-velocity of the observers at rest and show that the energy of the graviton as measured by the observers at rest is

     $$E = \frac{-p_0}{\sqrt{1 - 2M/r}} \,. \tag{2}$$

     Describe what the above formula implies as the graviton travels to larger and larger radii r.
   - As discussed in class, in geometric optics, if the waves travels precisely radially through the black-hole spacetime, then the amplitude of the wave fields decreases as $1/r$. Assume that these radially traveling waves are monochromatic and show that the gradient of their phase must have the form $\phi = \sigma(t - r_*)$ where $r_* = r + 2M \ln(r/(2M) - 1)$. [Hint: show that the gradient of this phase function is null and has $k_0 = p_0$ constant.
   - What is the energy E of a graviton for these waves, measured by the at-rest observer in terms of the constant σ? What is the frequency that the observer measures?
   - Combining the above results show that the radially traveling, monochromatic waves have the form

     $$h = \frac{A}{r} \cos[\sigma\,(r_* - t) + \delta] \,, \tag{3}$$

     where δ and A are constants.

2. **Neutron-star binary systems** (10 points)

   Consider a binary system composed of two $1.4 M_\odot$ neutron stars in a circular orbit.

   - If the orbital period is 7.75 h, how long will it be until the neutron stars collide?
   - Show that the rate of change of the orbital period is

$$\frac{dP}{dt} = -\frac{192\pi}{5} \frac{m_1 \, m_2}{(m_1 + m_2)^2} \left(\frac{2\pi G(m_1 + m_2)}{c^3 \, P} \right)^{5/3} \tag{4}$$

     where m_1 and m_2 are the neutron star masses, and compute dP/dt in $\mu\,\mathrm{s}\,\mathrm{yr}^{-1}$ when $P = 7.75\mathrm{h}$.
   - How far apart can the companions be if the time to collision is less than 10^{10} yr?
   - How much time remains before collision once the gravitational-wave emission frequency reaches 40 Hz? 100 Hz?
   - Assume the neutron star binary is at a distance $r = 1\mathrm{Mpc}$ and inclination $\theta = 0$ (i.e., face-on), plot with Mathematica or Maple the polarization $h_+(t)$ and the frequency $f(t)$ versus time for gravitational-wave frequencies between 100 and 300 Hz.