Homework: Week 2

Recommended readings:

1. Local Lorentz and free-falling frames, Sec. 8.4 in J. Hartle, "Gravity".
2. Proper detector frame, see, e.g., W. Ni and M. Zimmermann, Phys. Rev. D **17**, 1473 (1978).
3. Newtonian and relativistic tidal gravity, see e.g., Sec. 24.2-24.5 in R.D. Blandford and K.S. Thorne, http://www.pma.caltech.edu/Courses/ph136/ph136.html

Assignment to be turned in at the beginning of the class on Tuesday, February 12 by students registered to the course:

- State what of the above readings you have done
- Work the two exercises below

Exercises:

1. **Resonant mass detectors** (10 points)

   The first gravitational-wave detector was a resonant mass detector or bar detector. It was built at the University of Maryland by Joseph Weber in the late 60'. It was a large, heavy, metal bar. The bar would absorb an impinging gravitational wave and be set into oscillations. Hopefully, those oscillations would be detectable.

   The simplest way to model a bar detector is with a damped spring. Let us assume that we have to masses m_1 and m_2, along the x-axis, connected by a spring with spring constant k and subjected to a dissipative force $F_{\rm diss} = -b\, dx/dt$. In equilibrium the masses are separated by a length L. Let us assume that a plane gravitational wave with frequency ω arrives along the z-axis and it is polarized only along the x-axis, i.e., $h_+ = h\,\cos\omega\,t, h_\times = 0$.

   - Assuming that $L \ll \lambda_{\rm GW}$, with $\lambda = 2\pi c/\omega$, derive the equation of motion for the displacement of the masses $x(t)$ with respect to the equilibrium position. [Hint: follow the description in the local inertial frame.]
   - The solution of the equation of motion in the previous item can be written as $x(t) = A\,\cos(\omega\,t + \delta)$. Derive A and δ, and the maximum of the oscillation at resonance.
   - Derive the kinetic energy of the oscillations, the potential energy of the oscillator, the work done on the oscillator by the gravitational wave and the rate of energy dissipated. Compute those quantities averaging over a cycle of oscillations.
   - Assume that $h = 10^{-21}$, $L = 1\,$m, reduced mass $\mu = m_1\,m_2/(m_1+m_2) = 1000\,$kg, quality factor $Q = 10^{-6}$ and $f = \omega_0/2/\pi = 1\,$kHz, compute the maximum of oscillations at resonance and the total energy in the oscillations, i.e., kinetic energy and potential energy, averaging over a cycle of oscillations. When comparing the averaged total energy to the thermal energy at room temperature, what do you conclude?

2. **Attenuation of gravitational waves** (10 points)

   Assume that a gravitational wave encounters a viscous fluid, which is initially at rest with fluid four-velocity given by $u^\mu = (1, 0, 0, 0)$.

- The shearing of the fluid is described by the shear tensor

$$\sigma_{\mu\nu} = \frac{1}{2}\nabla_\mu u_\nu + \frac{1}{2}\nabla_\nu u_\mu + \frac{1}{2}u_\mu u^\alpha \nabla_\alpha u_\nu + \frac{1}{2}u_\nu u^\alpha \nabla_\alpha u_\mu - \frac{1}{3}(g_{\mu\nu} + u_\mu u_\nu)\nabla_\alpha u^\alpha \,. \tag{1}$$

  Show that the shear has purely spatial components when the gravitational-wave is expressed in the transverse-traceless (TT) gauge, and that

$$\sigma_{ij} = \frac{1}{2}\frac{\partial}{\partial t} h_{ij}^{\mathrm{TT}} \,. \tag{2}$$

- The shearing of the viscous fluid generates a contribution to the stress-energy tensor of the form

$$T_{\mu\nu} = -2\eta\, \sigma_{\mu\nu} \,, \tag{3}$$

  where we indicate with η the coefficient of viscosity of the fluid. What is the linearized field equation for the gravitational wave in the TT gauge in presence of the viscous fluid? Show that a plane wave travelling along the z-axis is attentuated by the fluid by an amount $e^{-z/l}$ where l is the attenuation length scale $l = c^3/(8\pi G\eta)$.

- Chocolate has a coefficient of viscosity of $\eta = 25\,\mathrm{kg}/(\mathrm{m\,s})$. Calculate the distance L that a gravitational wave must travel through chocolate before it is attentuated by a factor $1/e$.