Two-Fluid Model Surface Impedance: A Tutorial

Technical Report · July 2017

DOI: 10.13140/RG.2.2.28399.33441

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ABSTRACT

The two-fluid phenomenological treatment of the electrical conductivity of a superconductor leads to a frequency response with characteristics that can be described without referring to the quantum mechanical pairing mechanism. This provides a further treatment of the surface impedance. Microwave frequency and microwave current dependent measurements of surface impedance of thin and thick film cuprate superconductors is presented and explained in light of the two-fluid model. This tutorial serves as an introduction to the subject for students joining the Microwave Lab at Hope College.

1. Impedance

Resistance is usually understood to be the ratio of voltage to current, $R=V/I$. In electronic circuit theory however the concept becomes more complicated because inductance and capacitance need to be included in this ratio as well. It is then more appropriate to consider the more complete quantity of *impedance*, which is also the ratio of voltage to current, $Z=V/I$. Thus impedance too has units of Ohms.

Voltage and current are not in phase when capacitive or inductive elements are involved giving rise to the distinction between resistance and impedance. Impedance is a complex number with the real part being the resistance and the imaginary part being the *reactance*, $Z=R+j(X_L-X_C)$, with $j=\sqrt{-1}$. The reactance due to a capacitor depends on (angular) frequency $\omega(=2\pi f)$ and capacitance C as $X_C=1/\omega C$. The reactance due to an inductor L is $X_L=\omega L$.

Example A 100Ω resistor, a 1 pF (=10^{-12} Farads) capacitor, and a 1 nH (=10^{-9} Henry) inductor are all connected in series. What is the impedance at 1 GHz (=10^9 Hertz)?

$$Z = R + j(X_L - X_C)$$

$$Z = 100\ \Omega + j\left(\left(2\pi\times10^9\ \text{Hz}\right)\left(10^{-9}\ \text{H}\right) - \frac{1}{\left(2\pi\times10^9\ \text{Hz}\right)\left(10^{-12\ \text{F}}\right)}\right)$$

$$Z = 100 - j153\ \ \Omega \quad \therefore$$

If any of the reactive elements are in parallel, then their inverses combine the same way that resistors in parallel would combine.

Example The components from the previous example are connected in parallel? What is their impedance at 1 GHz?

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{jX_L} + \frac{1}{-jX_C} = 0.01 + j0.153\ \Omega^{-1}$$
$$Z = 0.43 - j6.5\ \Omega \quad \therefore$$

The usual method of inverting a complex number was used.

2. Surface Impedance

The surface of a conductor is treated as a circuit which contains resistors, capacitors and inductors. The surface then has a corresponding *surface impedance*, $Z_S=R_S+jX_S$, which is connected to conductivity σ through

$$Z_S = \frac{1+j}{\sigma\delta}, \tag{1}$$

where

$$\delta = \sqrt{\frac{2}{\omega\mu_o\sigma}} \tag{2}$$

is the skin depth[1]. μ_o=4πx10^{-7} T·m/A is the permeability of free space. The subscript S means *surface*. So X_S is the reactance of the surface, or the *surface reactance*.

Just as resistance in a circuit provides a measure of the conversion of electricity to heat, the real part of surface impedance, the surface resistance R_S, also provides a measure of electrical dissipation. The imaginary part of the surface impedance is the surface reactance X_S which provides a measure of field penetration (storage) into the surface, just as the reactance in a circuit provides a measure of electromagnetic energy storage. σ is a real number for a normal conductor, and thus $X_S=R_S$. However, superconductors have a complex conductivity, and X_S is consequently not equal to R_S.

Impedance is the ratio of voltage to current, $Z=V/I$, with units of Volts/Ampere, which is equivalent to an Ohm. Surface impedance is the ratio of electric field $E_{||}$ (units Volt/meter) parallel to a conducting surface to the magnetic field H (units Ampere/meter) at the surface. So the units of surface impedance, $Z_S=E_{||}/H$ are (Volts/meter)/(Amperes/meter), which is also

Ohms. Technically $Z_S = E_{||}/K$ where K (units Ampere/meter) is the surface current density. K and H are numerically equivalent at a conductor surface, just perpendicular in direction $\vec{K} = \hat{n} \times \vec{H}$, so the two ratios are identical. At non-zero frequency, the electric field drops off inside a conductor *as* $E = E_o e^{-z/\delta}$, where E_o is the electric field at the surface and z is the depth inside the surface.

Using $1 + j = \sqrt{2j}$, and combining Equation (1) and Equation (2) a useful expression for surface impedance is

$$Z_S = \sqrt{\frac{j\omega\mu_o}{\sigma}} \tag{3}$$

which appears to depend on the square root of frequency. Z_S also appears to be a complex number with equal real and imaginary parts. Both appearances are illusions. Surface impedance is a complex number which may have unequal real and imaginary parts, and it may not depend on frequency as $\omega^{1/2}$.

Because the conductivity of a normal metal is frequency independent at frequencies below the THz range, the surface impedance of normal metals does follow both of those expectations. The real and imaginary parts of Z_S are equal, and depend on frequency as $\omega^{0.5}$. Figure 1 shows the surface resistance of brass at room temperature over a wide range of microwave frequencies. Unlike normal conductors, superconductors have a frequency dependent complex conductivity, $\sigma(\omega)$, meaning that the surface impedance of a superconductor is a complicated expression of ω.

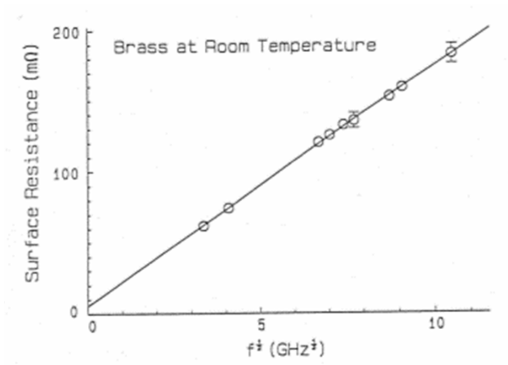

Figure 1. The surface resistance of brass is plotted against the square root of frequency[7] indicating that $R_S \propto \omega^{1/2}$.

3. The Two-Fluid Model

When a material is cooled below its superconducting transition temperature, T_C, it proceeds immediately into the superconducting phase. This is a *second order phase transition* because no latent heat is needed to complete the transition. Without latent heat, the material's temperature continues to drop without the pause associated with *first order phase transitions* such as freezing and boiling. Thus, the electrons in the material do not all at once jump into the superconducting phase. Rather, as the material cools below T_C the electrons become superconducting one-by-one, or rather pair-by-pair. As the material becomes colder, more electrons enter the superconducting phase. The total number density of electrons (i.e. number of electrons per cm^3) in the material is given by n, and the number density that is superconducting is given by n_s. How n_s depends on temperature is a matter of the physics of the superconductor. Using the empirical form for the temperature dependence[2]

$$n_s(T) = n_s(0)\sqrt{1-\left(\frac{T}{T_c}\right)^p} \tag{4}$$

where p=2 for high temperature cuprate superconductors which operate in what is called the *dirty limit*[3]. p=4 for conventional (low temperature) superconductors such as the various elemental metals and their alloys[4]. $n_S(0)/n$ is the fraction of electrons per unit volume that are superconducting at zero temperature, and is usually taken to equal unity.

Known as the *Gorter-Casimir* model[4], the presence of two types of electrons, coexisting in the superconductor can be described as two fluids of electrons. The use of the term "fluid" originates from the Drude treatment of electrons in metals where the charge carriers are treated as a fluid of electrons flowing through the conductor, much like water in a pipe[4]. So the two-fluid model of a superconductor treats each electron in the superconductor as participating in one of two fluids: the normal fluid n_n or the superconducting fluid n_s. These two fluids coexist, and they both flow simultaneously and in parallel through the superconductor. Each fluid has its own electrodynamic with its own skin depth and conductivity, but together, they form a composite electrodynamic because after all, it makes no sense for the electric field to have two different penetration distances and two different impedances.

Although there is always a skin effect associated with the screening response of normal electrons to time-dependent current, there is another depth, called the *London penetration depth* λ_L associated with the superelectrons screening any external magnetic field. The expulsion of magnetic field from the interior of a superconductor is known as the *Meissner-Ochsenfeld effect*[4]. Expulsion is accomplished by a surface anti-image current contained within λ_L of the surface. Usually λ_L is shorter than the skin depth δ and the superelectrons dominate the electrodynamic response. However, at temperatures close to T_C the skin depth is shorter and normal electrons dominate the electrodynamics. This makes the neighborhood of the transition temperature T_C an interesting temperature range for the high-frequency study of superconductors.

Solution to the Londons' (there were two Londons) constitutive relation provides the expression[4]

$$\lambda_L = \sqrt{\frac{m^*}{\mu_o n_s^* e^{*2}}} \tag{5}$$

where the asterisks require some explanation. Unknown in the days of the London Equation's derivation was that the superconducting fluid is comprised of electrons that form pairs[1]. So a single charge carrier in the superconducting fluid is a pair of electrons with mass $m^*=2m_e$, twice the electron mass, a charge $-e^*=-2e$, twice the electron charge, and a fluid density n_s^*, half the density of superconducting electrons. So, the asterisks are of no consequence in Equation (5) since the twos cancel.

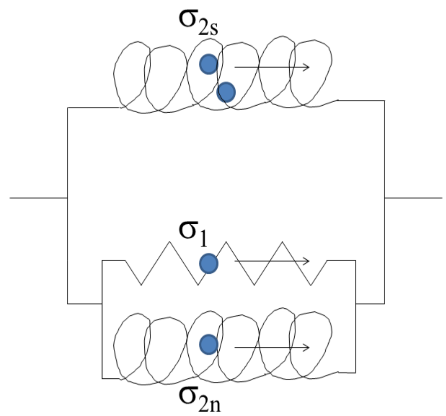

Figure 2. The equivalent circuit of a superconductor in the 2-fluid model. The superconducting electrons pass through an inductive path. The normal electrons pass through a resistive path, but because they are moving charge, they do have a small amount of inductance which will not be included in the derivation of total conductivity.

Since there are two "fluids" of carriers flowing in parallel through the superconductor at the same time, an equivalent circuit for a superconductor should look like Figure 2 where the normal fluid and the superelectron fluid are *shunting* each other. At zero frequency the inductive reactance of the imaginary part is zero and it short circuits (or shunts) the real resistive part, making superconductors lossless at DC, and only at DC. Derivation of the complex conductivity of a superconductor requires two things: the relationship between carrier speed and current, and a little classical mechanics.

4. Conductivity of a Superconductor

The current density *J* is the current passing through a cross-sectional area of conductor per unit cross-sectional area, and is related to the speed of the carriers through

$$\vec{J} = ne\vec{v} \qquad (6)$$

where *n* is the number of carriers per unit volume, *e* is the charge of each carrier and $\vec{v}$ is the drift velocity of the carriers. Equation (6) will be combined with some simple mechanics expressions and Ohm's law

$$\vec{J} = \sigma\vec{E} \qquad (7)$$

to give an expression for σ.

[1] That electrons form pairs is rather fundamental to superconductivity. The nature of that pairing mechanism is a matter of on-going research. That the pairing is of a d-wave nature in cuprate superconductors is well established, and supplants many of the formulations of BSC theory.

The superconducting electrons flow at velocity $\vec{v}_s$ without any damping, so Newton's second law applied to the motion of superconducting electrons is

$$m\vec{a} = \vec{F} \quad \text{or since } \vec{F} = (-e)\vec{E}$$
$$m\frac{d\vec{v}_s}{dt} = -e\vec{E}. \tag{8}$$

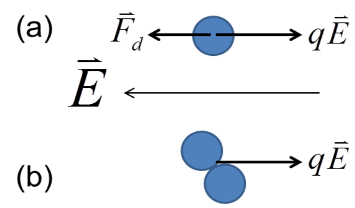

Figure 3. The free body diagrams for (a) normal electrons and (b) superconducting charge carriers. Recall that the drag force is in the opposite direction as motion and that the force on an electron is opposite the direction of the electric field.

based on the free body diagram in Figure 3b. The motion of the normal electrons at velocity $\vec{v}_n$ is damped by collisions with atoms with a drag force that is proportional to the speed of the normal electrons and their mass. The drag force is also inversely proportional to the time, τ, between collisions between the electrons and the atoms, so altogether

$$\vec{F}_{drag} = -m\frac{\vec{v}_n}{\tau}. \tag{9}$$

Think about it, the more frequently electrons encounter atoms, the smaller τ is and the more drag force there will be. Newton's second law applied to the motion of the normal electrons is then

$$m\frac{d\vec{v}_n}{dt} = -m\frac{\vec{v}_n}{\tau} - e\vec{E} \tag{10}$$

based on the free body diagram in Figure 3a.

It is customary in experiments to apply AC signals that vary harmonically with time. That is, the electric field varies sinusoidally with time, and in complex form,

$$\vec{E}(t) = \vec{E}_o e^{-j\omega t}. \tag{11}$$

Since current responds to applied voltage or electric field, J will have the same time dependence

$$\vec{J}_s(t) = \vec{J}_{so} e^{-j\omega t} \quad \text{and} \quad \vec{J}_n(t) = \vec{J}_{no} e^{-j\omega t}. \tag{12}$$

where J_{so} and J_{no} are the amplitudes of the current of superconducting electrons and the current of normal electrons, respectively. Using $v=J/ne$ for carrier velocity and inserting Equations (11) and (12) into Equations (8) and (10), gives

$$\vec{J}_{so} = -j\frac{e^2 n_s}{\omega m}\vec{E}_o \quad \text{and}$$
$$\vec{J}_{no} = \frac{e}{j\omega\frac{m}{n_n e} + \frac{m}{n_n e\tau}}\vec{E}_o.$$

Combining these two expressions into $\vec{J}_{so} + \vec{J}_{no} = \sigma\vec{E}_o$ produces, after algebraic manipulation to get the complex expression out of the denominator, an expression for the complex conductivity of a superconductor

$$\vec{J}_{so} + \vec{J}_{no} = \left(\frac{m}{n_n\tau\left[\left(\frac{m\omega}{n_n e}\right)^2 + \left(\frac{m}{n_n e\tau}\right)^2\right]} - j\left(\frac{e^2 n_s}{m\omega} + \frac{m\omega / n_n}{\left(\frac{m\omega}{n_n e}\right)^2 + \left(\frac{m}{n_n e\tau}\right)^2}\right)\right)\vec{E}_o \tag{13}$$

The quantity in parentheses in Equation (13) is the complex conductivity, σ. After further simplification it reduces to

$$\sigma = \frac{n_n e^2\tau}{m(1+\omega^2\tau^2)} - j\left(\frac{n_s e^2}{m\omega} + \frac{n_n e^2\omega\tau^2}{m(1+\omega^2\tau^2)}\right) \tag{14}$$

revealing simple expressions for the real and imaginary parts of the complex conductivity.

Unless frequency is very high (infrared maybe), $(\omega\tau)^2 \ll 1$, and Equation (14) reduces to

$$\sigma \approx \frac{n_n e^2\tau}{m} - j\frac{n_s e^2}{m\omega} \quad \text{or simply}$$
$$\sigma \approx \sigma_1 - j\sigma_2. \tag{15}$$

Although arguably in final form, Equation (15) is subject to further illumination by including elements from the theory of metals. The normal state conductivity comes from the Drude model and is

$$\sigma_n = \frac{ne^2\tau}{m}. \tag{16}$$

Not all of the electrons are in the normal state and the real part of σ is attenuated by the suppressed normal carrier density below T_C. Using this argument to rewrite Equation (15) using Equations (5) and (16) results in the most commonly cited form of the conductivity

$$\sigma \approx \frac{n_n}{n}\sigma_n - j\frac{1}{\omega\mu_o\lambda^2}$$
$$\sigma \approx \sigma_1 - j\sigma_2 \tag{17}$$

where λ may not actually be the London penetration depth as the actual penetration depth is usually larger do to the granularity of the material. Hence it is advisable that we dispense with the subscript L on λ_L. σ_1 and σ_2 are simply referred to as the real and imaginary parts of the conductivity, respectively. Equation 17 can be referenced for simple expressions of σ_1 ($=n_n\sigma_n/n$) and σ_2 ($=1/\omega\mu_o\lambda^2$) based on material properties. The real and imaginary parts depend on carrier concentration n, either directly as in the case of σ_1, or through the dependence of λ on n[5]. Since T_C also depends on carrier concentration[6], changes in the doping level simultaneously affects numerous superconducting properties: loss, field penetration, and T_C.

5. Surface Impedance from the Two-Fluid Model

The surface impedance can now be written using Equation (17)

$$Z_S = \sqrt{\frac{j\omega\mu_o}{\sigma}} = \sqrt{\frac{j\omega\mu_o}{\sigma_1 - j\sigma_2}}\,. \tag{18}$$

A useful superconductor has very little electrical loss, which means that the real, or resistive, part of the conductivity is very small compared to the imaginary, or inductive part, so $\sigma_1 << \sigma_2$ and moving the complex expression into the numerator Equation (18) is rewritten as

$$Z_S \approx \sqrt{\frac{\omega\mu_o}{\sigma_2}}\sqrt{j\frac{\sigma_1}{\sigma_2} - 1}\,. \tag{19}$$

Using the binomial expansion theorem which says that $(1+x)^n \approx 1+x/n$ *if* $x<<1$, Equation (19) becomes

$$Z_S \approx \sqrt{\frac{\omega\mu_o}{\sigma_2}}\left(j + \frac{\sigma_1}{2\sigma_2}\right). \tag{20}$$

Now, combine Equations (17) and (20) to eliminate σ_1 and σ_2 to get

$$Z_S \approx \sqrt{\frac{\omega^2\mu_o m}{n_s e^2}}\left(j + \frac{1}{2}\frac{n_n\sigma_n\omega\mu_o\lambda^2}{n}\right) \tag{21}$$

where recall that

$$\lambda = \sqrt{\frac{m}{\mu_o n_s e^2}} \ . \tag{22}$$

So using this and switching the order of the real and imaginary parts,

$$Z_S = \frac{1}{2}\frac{n_n}{n}\sigma_n \omega^2 \mu_o^2 \lambda^3 + j\omega\mu_o\lambda = R_S + jX_S \tag{23}$$

is the most useful expression for the surface impedance based on the two-fluid model.

Frequency Dependence of the Surface Impedance

The London penetration depth does not depend on frequency (e.g. superconductors are *non-dispersive*). The surface resistance of a superconductor therefore according to Equation 23 depends on frequency squared, as opposed to the square root of frequency for normal conductors. Notice that the surface reactance depends linearly on frequency. Deviations from quadratic frequency dependence of R_S, to as weak as linear frequency dependence, have been observed for granular superconducting materials indicating that λ does have some frequency dependence[7]. This is because granularity introduces some contribution of the normal fluid to the electrodynamics and λ is thus not entirely equal to the London penetration depth[8]. This is not likely the case for high quality superconducting thin films, as seen in Figure 4[9], which maintain the quadratic frequency dependence up to at least 100 GHz. This figure also includes comparison to brass and to copper at 77 K, which have square root frequency dependence. Notice that above 30 GHz, the superconductor actually has higher R_S than copper. 30 GHz is then referred to as the *cross-over frequency* for this superconducting sample.

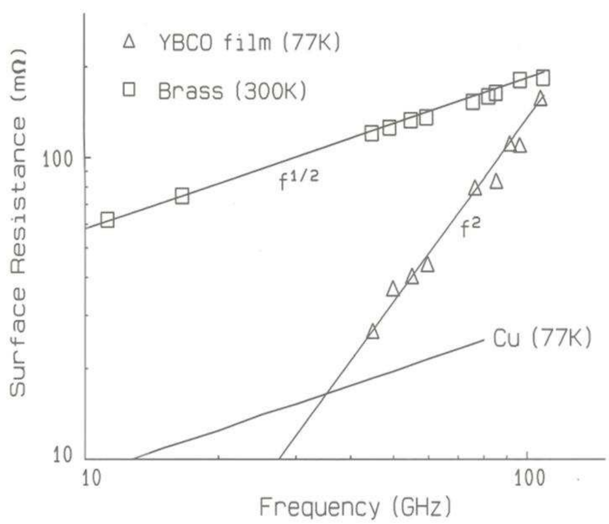

Figure 4 The frequency dependence of the surface resistance of a YBCO thin film measured at 77K using an open Fabry-Perot resonator. Comparison to normal metals, copper and brass, is included[9].

Temperature Dependence of the Surface Impedance

Although the frequency dependence of surface impedance is brilliantly confirmed by experiment, the temperature dependence is in not such good agreement. Combining Equations (4), (5) and the real part of Equation (21) gives

$$R_S = \frac{\sigma_n \omega^2 \mu_o^3 \lambda_0^3}{2} \left[\frac{1 - \sqrt{1 - \left(\frac{T}{T_C}\right)}}{\left(1 - \left(\frac{T}{T_C}\right)^2\right)^{3/2}} \right]. \tag{24}$$

There are two issues. First, the two-fluid model predicts that R_S should go to zero at zero temperature, and it doesn't. Instead it levels off at what has come to be called the *residual surface resistance*. Second, the two-fluid model predicts that the surface impedance becomes infinite at T_C, which also does not happen. Instead as T_C is approached, R_S arrives at its "above T_C value", sometimes called the *classical surface resistance*. At temperatures not near zero or T_C, Equation (24) provides a better fit. This has made the two temperature extremes of the superconducting state fertile ground for research. Figure 5 shows the temperature dependence of a granular thick film of YBCO measured at 11.28 GHz[9]. Below about 70 K, the R_S has leveled off. And it certainly does not become infinite at its T_C of 92 K. Also shown is the effect of a perpendicular static magnetic field which raises the R_S by decoupling the superconducting grains.

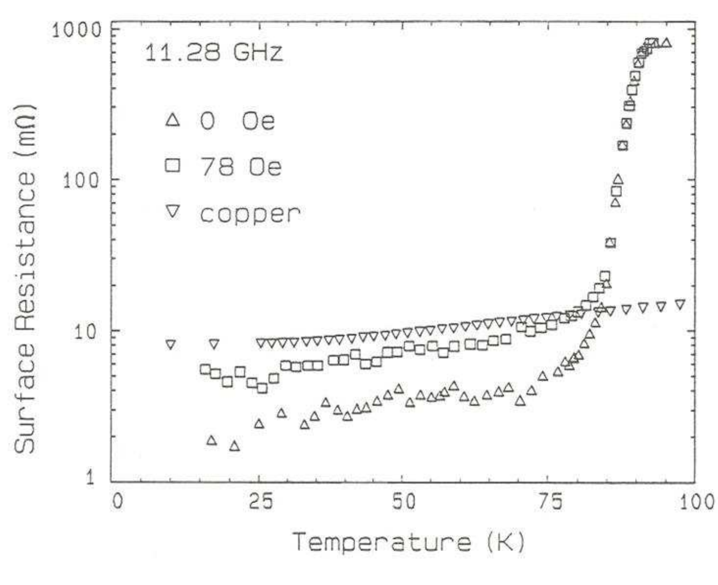

Figure 5 The temperature dependence of a granular thick film of superconducting YBCO. Also shown is the effect of a static magnetic field applied perpendicular to the film's surface[9].

Microwave Current Dependence of the Surface Impedance

The surface impedance of a superconductor is notoriously dependent on the microwave current above a certain level called the *nonlinearity onset current*. Inspection of Equation (23) shows where the current dependence enters into the surface impedance. The imaginary term only contains three quantities: ω, μ_o and λ. Of these three quantities, the first two do not have anything to do with current. Frequency is prescribed by the source and μ_o is a fundamental constant. So the surface reactance, X_S, is current dependent because the penetration depth is current dependent. Any change in λ due to microwave current happens in equal measure to X_S.

The surface resistance contains two quantities that vary with current: n_n and λ. The dependence of these quantities on current is complicated and reveals much about the physics of the superconductor. λ and n_n change for a variety of reasons. Flux penetration into the superconductor, local microwave heating, and grain boundary decoupling are among the phenomena that change penetration and carrier density (also called the "order parameter"). What is unsettling here is that the RF magnetic field H_{RF} is used to measure Z_S. So a small value of H_{RF} is needed to ensure that Z_S is not influenced by its measurement. However, the influence of H_{RF} on Z_S, shown in Figure 6, is a matter of research in its own right[10].

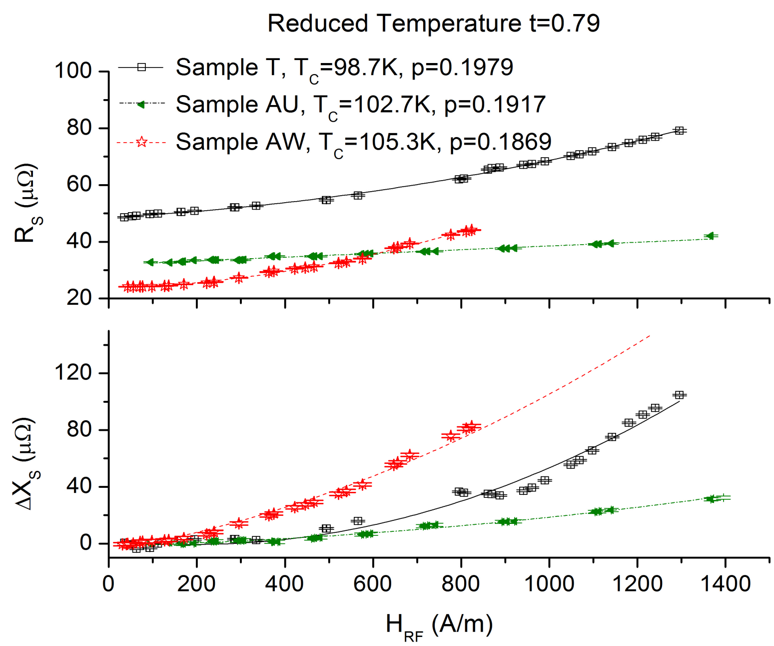

Figure 6 The surface impedance is influenced by the RF current that is used to measure it, as seen in these three superconducting TBCCO thin films measured with a sapphire dielectric resonator. *p* refers to the carrier doping level of the sample[10].

6. Conclusions

The two-fluid model provides a physical description of the condensation of electrons into the superconducting state. It makes various predictions about how the surface resistance and surface reactance should depend on frequency, temperature and current. When there are arguably no dominant morphological features, such as granularity, contributing to the surface impedance, then the two-fluid model's dependencies are observed. Granularity however masks the behavior of Z_S. Elevated RF current, which is used to probe the Z_S, breaks charge carrier pairs and thus influences Z_S. This *nonlinearity* has broader implications in device performance as it distorts the RF signal, producing harmonic and intermodulation mixing tones[11].

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