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VORTICES IN COMPLEX SCALAR FIELDS

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Consider a spatially extended field which evolves in time according to a PDE. The solutions contain particle-like defects whose motions are parts of the full dynamics. We inquire into the formulation of an asymptotic "particle + field" defect dynamics which derives from the full PDE. We carry out this program for two complex scalar field equations in two space dimensions, the nonlinear Schrödinger equation and the nonlinear heat equation. The topological defects are zeros of the complex scalar field with nonzero integer winding numbers, called *vortices*. Vortices evolving under the nonlinear Schrödinger equation behave like point vortices in ideal fluid. Pairs of vortices evolving under the nonlinear heat equation with like (opposite) winding numbers undergo a repulsive (attractive) interaction.

1. Introduction

Let $\psi(\boldsymbol{x}, t)$ be a complex scalar field defined for $\boldsymbol{x} \in \mathbb{R}^2$ and some interval of time t. Its time evolution is governed by a PDE in variables $\boldsymbol{x}$ and t. Assume that at $t = 0$, ψ has isolated zeros with nonzero integer winding numbers. As $\psi(\boldsymbol{x}, t)$ changes continuously in time, these zeros persist as long as they remain isolated. They retain their original winding numbers but may migrate in space. We call these zeros of ψ *vortices*.

A solution of the PDE naturally encodes the trajectories of the vortices and in this sense the vortices seem secondary in importance – "merely local structures in an overall flow". But clearly they are dominant features in the geometry and topology of the field and it is natural to expect situations in which they "dominate" the dynamics as well. We formulate a basic scenario for "vortex dominated" evolutions of the field ψ. The PDE contains fundamental length and time scales. Relative to these, the vortices are far apart and move slowly. The field far from the vortices varies slowly in space and time. In this slowly varying limit we derive a reduced PDE from the full PDE. The slowly varying field description breaks down within a fundamental length of a vortex and we require local descriptions of the field about the vortices. Asymptotic matching leads to effective boundary conditions on the slowly varying field at a vortex. The reduced PDE governing the slowly varying field subject to the effective boundary conditions at vortices constitutes an "asymptotic vortex dynamics". Such formulations are reminiscent of "particle + field" theories well known in physics. In the simplest case, the asymptotic vortex dynamics reduces to a set of ordinary differential equations for the positions of the vortices.

We actually carry out this program for two closely related field equations, both well known in mathematical physics. They are

$$\Delta\psi + (1 - |\psi|^2)\psi = -\mathrm{i}\psi_t \quad \text{(NLSE)} \tag{1.1}$$

$$\Delta\psi + (1 - |\psi|^2)\psi = \psi_t \quad \text{(NLHE)}. \tag{1.2}$$

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The equations (1.1), (1.2) are known as the nonlinear Schrödinger equation (NLSE) and the nonlinear heat equation (NLHE), respectively.

1.1. Vortex states

In the time-independent case, both (1.1) and (1.2) reduce to

$$\Delta\psi + (1 - |\psi|^2)\psi = 0. \tag{1.3}$$

The simplest solutions of (1.3) are the *uniform states*

$$\psi = e^{i\theta_0}, \tag{1.4}$$

where θ_0 is an arbitrary constant. These are stable as solutions of the NLSE or NLHE. In addition there are *vortex solutions* with a single winding number n, which take the form

$$\psi(x) = U(r)\exp[i(n\theta + \theta_0)] \tag{1.5}$$

where (r, θ) are polar coordinates of $\mathbb{R}^2$ and θ_0 is an arbitrary constant. The modulus $U(r)$ satisfies the boundary value problem

$$U_{rr} + \frac{1}{r}U_r - \frac{n^2}{r^2}U + (1 - U^2)U = 0, \tag{1.6}$$

in $r > 0$, and

$$U(0) = 0, \qquad U(\infty) = 1. \tag{1.7}$$

The continuity of ψ at $r = 0$ forces $U(0) = 0$, while $U(\infty) = 1$ is consistent with a locally uniform state as $r \to \infty$. Asymptotic behaviors of $U(r)$ as $r \to 0$ or $r \to \infty$ may be established directly from (1.6), (1.7):

$$\begin{aligned} U(r) &\sim ar^{|n|} + \mathscr{O}(r^{|n|+2}) && \text{as } r \to 0, \\ &\sim 1 - n^2/2r^2 + \mathscr{O}(1/r^4) && \text{as } r \to \infty. \end{aligned} \tag{1.8}$$

Fig. 1 depicts numerical solutions of $U(r)$ for $n = 1 \to 4$. From these, we may construct corresponding visualizations of $\psi(x)$. These are presented in fig. 2. The neighborhood of $r = 0$ where U is significantly less than one is called the vortex core. Its radius, the *core radius*, defines a characteristic length. We see that the core radius increases with n.

Vortices with winding numbers $n = +1$ or -1 are *topologically* stable, while vortices whose winding numbers have absolute value 2 or greater are topologically unstable against fission into $|n|$ vortices, each of winding number sgn n. There is strong numerical evidence that the topological stability of vortices indicates their *dynamical* stability as solutions of the NLHE. In numerical simulations of interactions between $|n| = 1$ vortices governed by the NLHE [1], the ψ field in a given vortex core retains its characteristic "vortex structure" as long as the vortex remains separated from its neighbors by a distance greater than a core radius. This is strong evidence for the dynamical stability of $|n| = 1$ vortices. The numerical simulations of Carlson and Miller also demonstrate the dynamical instability of $|n| \geq 2$

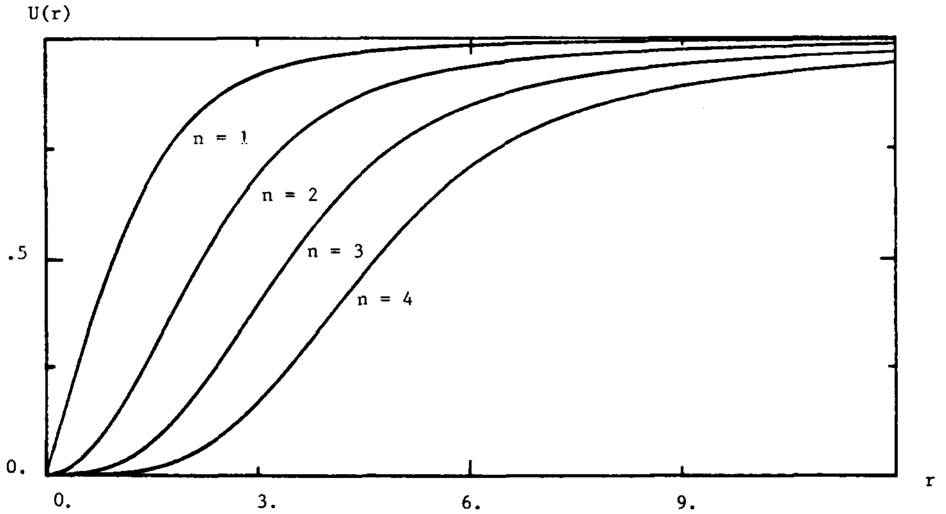

Fig. 1

vortices against fission into $|n|$ vortices, each of winding number sgn n. Fig. 3a is a movie of an $n=2$ vortex disintegrating into two $n=1$ vortices. In fig. 3b, we observe the initial disintegration of an $n=3$ vortex into four $n=+1$ vortices and a single $n=-1$ vortex. The $n=-1$ vortex eventually merges with one of the $n=+1$ vortices and both are annihilated. So finally, only three $n=+1$ vortices remain.

The dynamical stability of vortices as solutions of the NLSE remains an open problem. The nondissipative character of the NLSE greatly complicates the stability theory and direct numerical simulation. In this work, we proceed on the assumption that $|n|=1$ vortices are dynamically stable, so that the problem of their interactions makes sense.

1.2. Asymptotic vortex dynamics

We describe the limit process which underlies the "asymptotic vortex dynamics" of the NLSE or NLHE. The intervortex distances are $\mathcal{O}(1/\epsilon)$, $0<\epsilon\ll 1$. The NLSE and NLHE are both first order in time and second order in space. Hence the natural time scale corresponding to the length scale $1/\epsilon$ is $1/\epsilon^2$. The natural representation of a vortex trajectory is

$$X=Q(T,\epsilon), \tag{1.9}$$

where X and T are scaled space and time variables

$$X=\epsilon x, \qquad T\equiv\epsilon^2 t. \tag{1.10}$$

The velocity associated with the trajectory (1.9) is $\epsilon\dot{Q}$, so the vortex velocities are $\mathcal{O}(\epsilon)$.

We construct complimentary asymptotic expansions of $\psi(x,t,\epsilon)$. The *core* expansions are valid in neighborhoods of radii $\mathcal{O}(1)$ about the zeros of ψ. The core expansion about $x=Q(T,\epsilon)/\epsilon$ takes the form

$$\psi=\psi(r,T,\epsilon), \qquad r\equiv x-Q(T,\epsilon)/\epsilon. \tag{1.11}$$

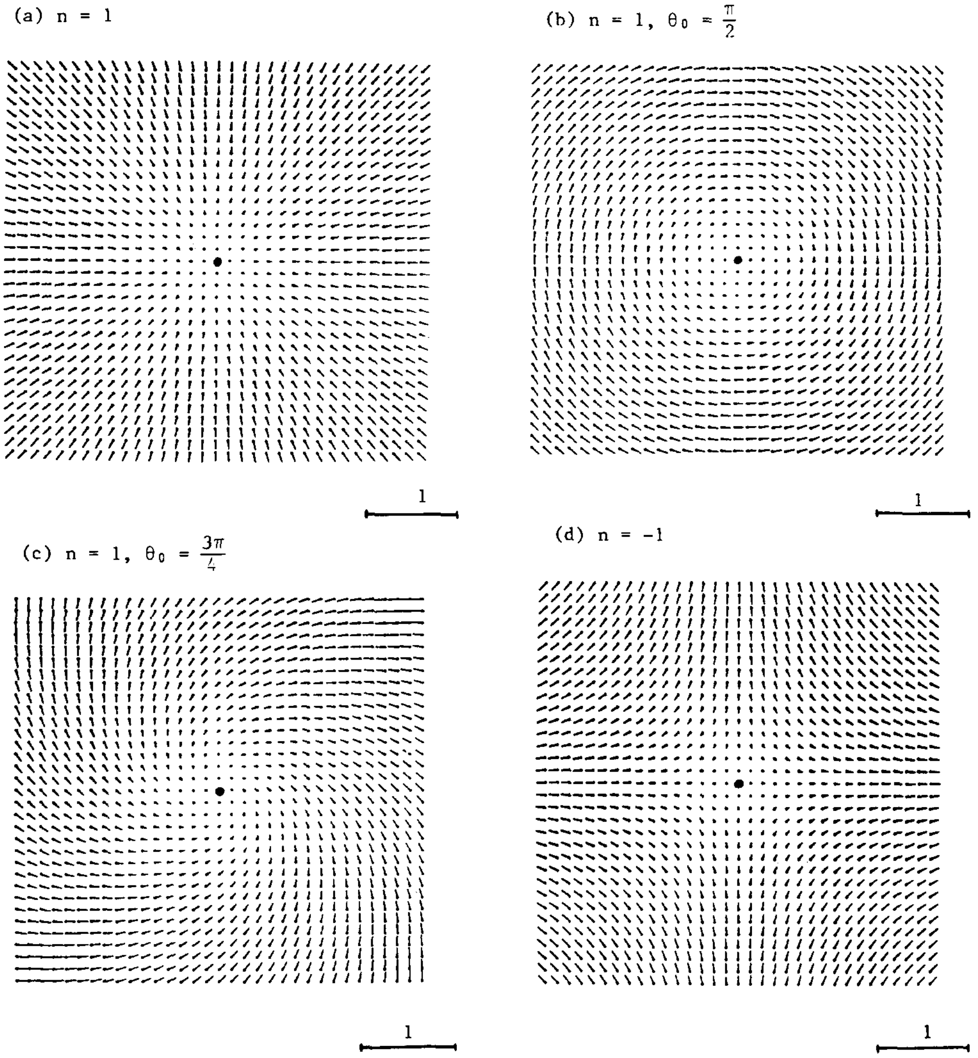

Fig. 2. In these visualizations, the dots mark a lattice of positions in the x, y plane. The ruler at the lower right corner of each graph indicates the spatial scale. The line segment emanating from each lattice position (x, y) represents the vector $(\mathrm{Re}\,\psi, \mathrm{Im}\,\psi)$ at (x, y). The value of the phase constant θ_0 is posted only if it is nonzero.

The *far field* expansion is valid at distances $\mathscr{O}(1/\epsilon)$ from a vortex. It takes the form

$$\psi = \Psi(X, T, \epsilon). \tag{1.12}$$

We describe certain subtle points about the ϵ-dependences of these expansions. They are constructed assuming that the vortex trajectories $Q(T, \epsilon)$ are *given*. Hence, they have *implicit* ϵ-dependences due to their dependences upon the $Q(T, \epsilon)$. Vortex evolutions are inherently 2D phenomena. One essential 2D effect is the appearance of $\log \epsilon$ as a parameter in the expansions. A logarithmic dependence upon ϵ is "slower" than any positive power α of ϵ in that $\epsilon^\alpha \log \epsilon \to 0$ as $\epsilon \to 0$. We treat unity and $\log \epsilon$ as formally equal orders of magnitude. In summary, we construct the core and far field expansions as formal

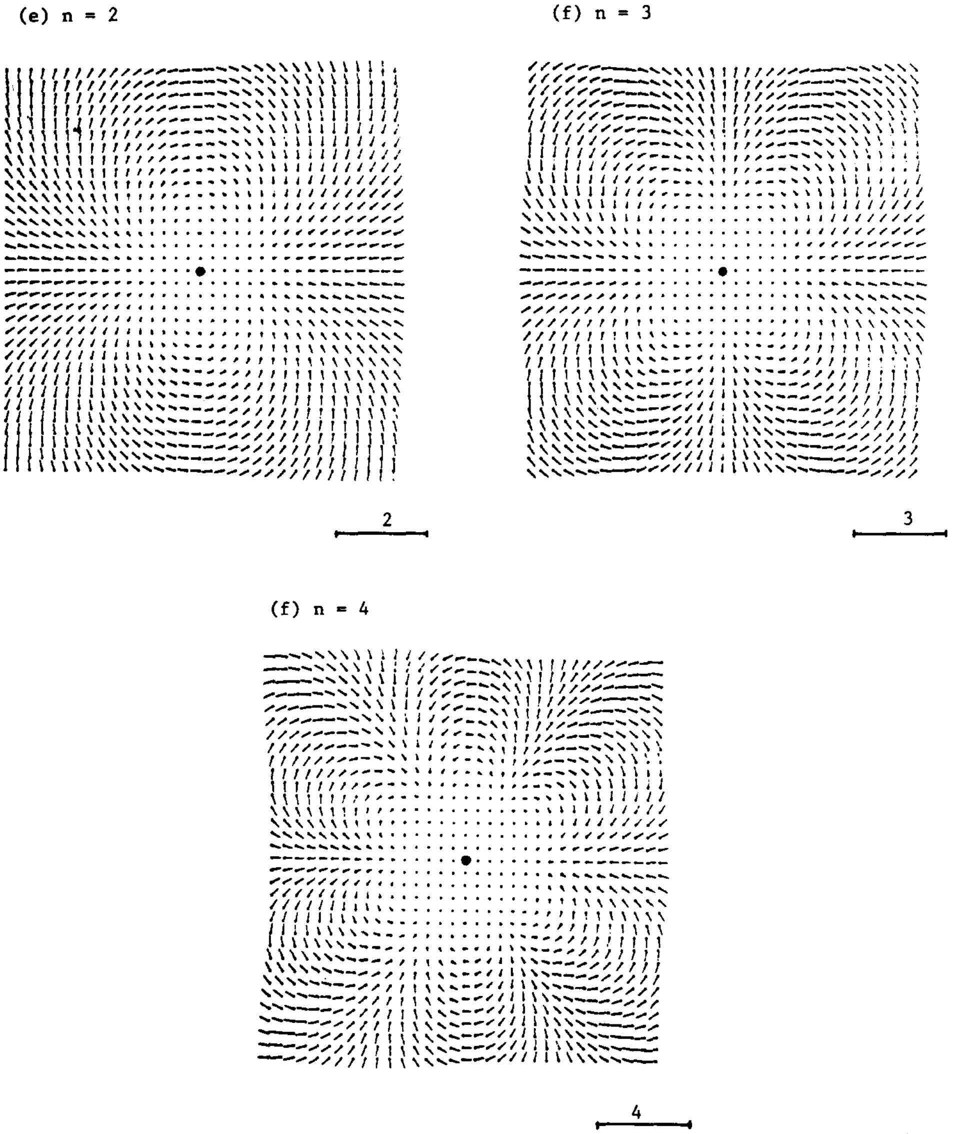

Fig. 2. Continued.

power series in ϵ, where each term in these expansions may exhibit implicit ϵ-dependence through dependence upon the $Q(T,\epsilon)$, and "slow" ϵ-dependence through dependence upon $\log \epsilon$.

Since the vortices are separated by $\mathcal{O}(1/\epsilon)$ distances and move with $\mathcal{O}(\epsilon)$ velocities, we anticipate that the leading order core expansion ψ^0 is a vortex state

$$\psi^0 = U(r)\exp\{\mathrm{i}[n\theta + \theta_0(T,\epsilon)]\}. \tag{1.13}$$

Here, r, θ are polar coordinates of $\boldsymbol{x}$. The winding number n is assumed to be $+1$ or -1 for topological and dynamical stability, and $\theta_0(T,\epsilon)$ is a slowly varying phase shift. The leading order far field expansion

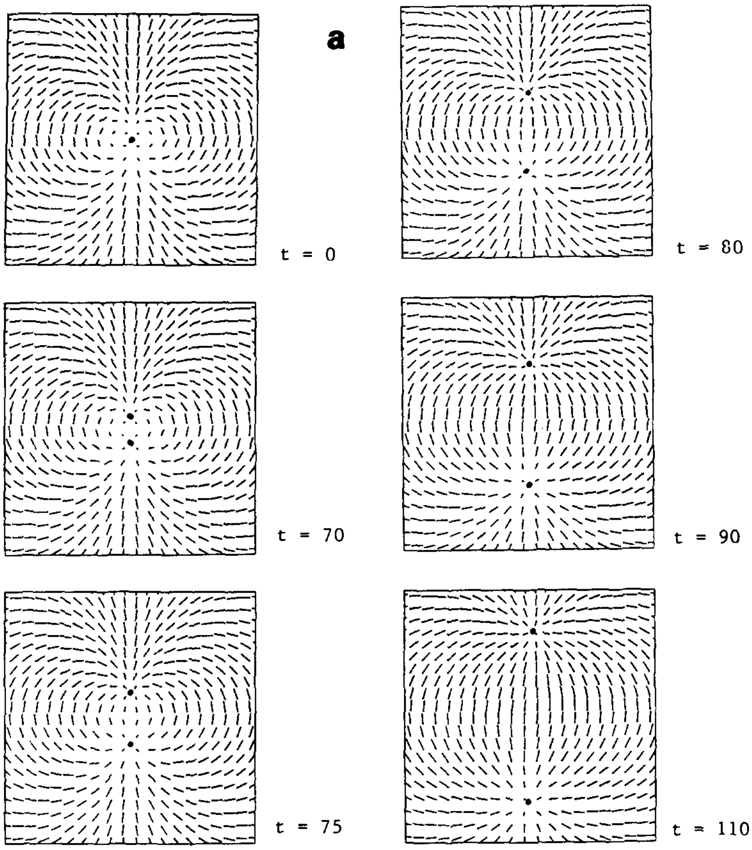

Fig. 3

Ψ^0 has modulus one. Hence there is a slowly varying phase $\Theta^0(X, T, \epsilon)$ so that

$$\Psi^0 = \exp[i\Theta^0(X, T, \epsilon)]. \tag{1.14}$$

The program of the asymptotic analysis is to derive a "free boundary problem" which governs the evolution of the slowly varying phase $\Theta^0(X, T, \epsilon)$ and the vortex positions $Q(T, \epsilon)$ in the limit $\epsilon \to 0$. This consists of a reduced PDE for the phase subject to effective boundary conditions at vortices. The effective boundary conditions emerge as requirements for *asymptotic matching* of the core and far field expansions. Fig. 4 summarizes in visual form the limit process associated with the asymptotic vortex dynamics.

1.3. Vortex dynamics of the NLSE

It follows from the NLSE that the far field phase Θ^0 satisfies Laplace's equation,

$$\Delta\Theta^0 = 0. \tag{1.15}$$

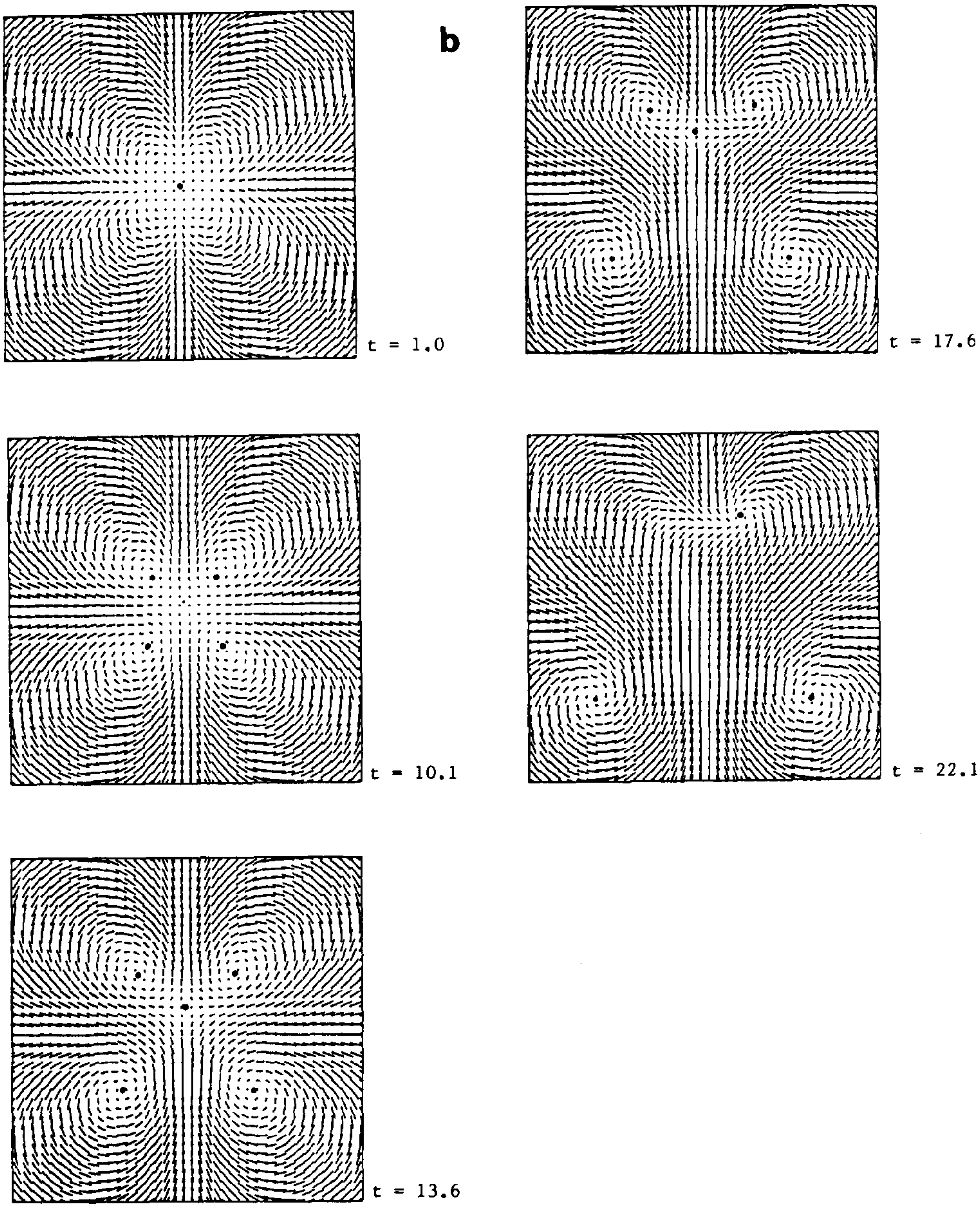

Fig. 3. Continued.

We describe effective boundary conditions upon Θ^0 at vortex positions determined from asymptotic matching.

(i) Topological boundary conditions

At each vortex position $\boldsymbol{X} = \boldsymbol{Q}$, Θ^0 satisfies the *topological* boundary condition

$$\Theta^0 \sim n\theta(\boldsymbol{R}) + \theta_0 \tag{1.16}$$

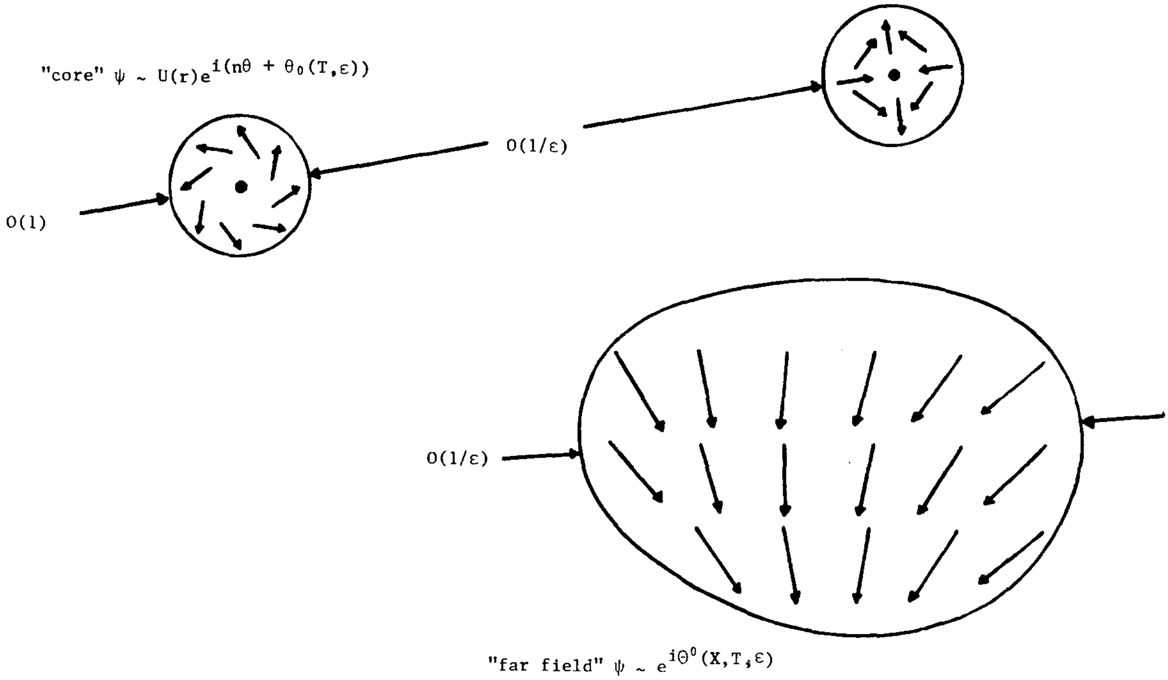

Fig. 4

as $\boldsymbol{R} \equiv \boldsymbol{X}-\boldsymbol{Q} \to 0$. Here, $\theta(\boldsymbol{R})$ denotes the azimuthal angle of the two-vector $\boldsymbol{R}$. $\theta_0 = \theta_0(T, \epsilon)$ is the same slowly varying phase shift which appears in the leading order core expansion (1.13). (1.16) forces the phase field Θ^0 to be multivalued, but $\Psi^0 = e^{i\Theta^0}$ remains single valued. Let C be any closed loop which contains only one vortex at $\boldsymbol{X}=\boldsymbol{Q}$ in its interior. The continuity of $e^{i\Theta^0}$ in a punctured neighborhood of $\boldsymbol{X}=\boldsymbol{Q}$ imposes upon Θ^0 the topological constraint that $(1/2\pi)\int_C \nabla\Theta^0 \cdot \mathrm{d}\boldsymbol{x}$ be an integer. From (1.16) it follows that the integer is the winding number n of the vortex at $\boldsymbol{X}=\boldsymbol{Q}$.

Laplace's equation (1.15) subject to the topological boundary conditions at vortices and a suitable boundary condition as $|\boldsymbol{x}| \to \infty$ is sufficient to determine Θ^0 if the vortex positions and winding numbers are given. For instance, suppose there are N vortices at positions $\boldsymbol{X}=\boldsymbol{Q}_i$, $i=1 \to N$, with winding numbers n_i. We take the boundary condition at ∞ to be $|\nabla\Theta^0| \to 0$ as $|\boldsymbol{x}| \to \infty$. It follows that the solution for Θ^0 is

$$\Theta^0 = \sum_{i=1}^{N} n_i \theta(\boldsymbol{X}-\boldsymbol{Q}_i) + C(T), \tag{1.17}$$

where $C(T)$ is a function of the slow time T.

(*ii*) *Dynamical boundary condition*

The *dynamical* boundary condition at $\boldsymbol{X}=\boldsymbol{Q}$ determines the vortex velocity $\dot{\boldsymbol{Q}}$ induced by the local structure of the phase field Θ^0 surrounding $\boldsymbol{X}=\boldsymbol{Q}$. In a region $\mathrm{D} \subset \mathbb{R}^2$ containing only one vortex at $\boldsymbol{X}=\boldsymbol{Q}$, the solution of Laplace's equation (1.15) subject to the topological boundary condition (1.16) takes the form

$$\Theta^0 = n\theta(\boldsymbol{R}) + H(\boldsymbol{X}, T, \epsilon), \tag{1.18}$$

where H is harmonic in D. From (1.18) we deduce that

$$\Theta^0 \sim n\theta(\boldsymbol{R}) + \theta_0 + \boldsymbol{K}\cdot\boldsymbol{R} + \mathscr{O}(R^2) \tag{1.19}$$

as $R \to 0$. Here, $\theta_0 = \theta_0(T,\epsilon) \equiv H(\boldsymbol{Q}, T, \epsilon)$, and $\boldsymbol{K} = \boldsymbol{K}(T,\epsilon) \equiv \nabla H(\boldsymbol{Q}, T, \epsilon)$. We think of $\boldsymbol{K}$ as the "locally uniform component" of the phase gradient at $\boldsymbol{X} = \boldsymbol{Q}$. It determines the vortex velocity: The dynamical boundary condition at Θ^0 at $\boldsymbol{X} = \boldsymbol{Q}$ is

$$\dot{\boldsymbol{Q}} = -2\boldsymbol{K} + o(1). \tag{1.20}$$

The Laplace equation (1.15) subject to topological and dynamic boundary conditions (1.16), (1.20) constitute the asymptotic vortex dynamics of the NLSE. Given a prescribed boundary condition on Θ^0 as $|\boldsymbol{X}| \to \infty$ and initial vortex positions, we can evolve the vortex positions and phase field Θ^0 by means of eqs. (1.15), (1.16), (1.20). For a system of N vortices in unbounded $\mathbb{R}^2$ with $|\nabla\Theta^0| \to 0$ as $|\boldsymbol{X}| \to \infty$, this formulation quickly reduces to a system of ODEs for the vortex positions. Let $\boldsymbol{X} = \boldsymbol{Q}_i$, $i = 1 \to N$ denote the vortex positions. Θ^0 is given by (1.17). Taking $\boldsymbol{Q} = \boldsymbol{Q}_i$ in (1.18), we find that (1.17) is consistent with (1.18) if

$$H = \sum_{j\neq i} n_j \theta(\boldsymbol{X} - \boldsymbol{Q}_j). \tag{1.21}$$

We compute the locally uniform component of phase gradient at $\boldsymbol{X} = \boldsymbol{Q}_i$ to be

$$\boldsymbol{K} = \nabla H(\boldsymbol{Q}_i, T, \epsilon) = \sum_{j\neq i} n_j \frac{J(\boldsymbol{Q}_i - \boldsymbol{Q}_j)}{|\boldsymbol{Q}_i - \boldsymbol{Q}_j|^2},$$

where

$$J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the matrix representing rotation by $\pi/2$ counterclockwise radians. The dynamical boundary condition (1.20) now implies

$$\dot{\boldsymbol{Q}}_i = 2\sum_{j\neq i} n_j \frac{J(\boldsymbol{Q}_i - \boldsymbol{Q}_j)}{|\boldsymbol{Q}_i - \boldsymbol{Q}_j|^2} + o(1). \tag{1.22}$$

1.4. *Fluid dynamics analogy*

Eqs. (1.19) are equivalent to the well-known Kirchoff equations describing the motion of point vortices in ideal incompressible fluid provided that we identify $\Gamma_i = 4\pi n_i$ as the circulation of the vortex at $\boldsymbol{X} = \boldsymbol{Q}_i$. This description of nonlinear Schrödinger vortex dynamics is quite natural when we recall a well-known analogy between the NLSE and the equations of motion for potential flow of ideal fluid [2]. In fact, the association between Kirchoff's equations and the vortex dynamics of the NLSE was first conjectured by means of the fluid dynamics analogy [3]. In the fluid dynamics analogy, $2 \arg \psi$ plays the role of the velocity potential. The $\epsilon \to 0$ far field limit process of the asymptotic vortex dynamics is analogous to an incompressible limit of the fluid dynamics. In the incompressible limit the velocity potential is harmonic. This is upheld in our analogy: In the far field, $2 \arg \psi \to 2\Theta^0$ as $\epsilon \to 0$, where Θ^0 is harmonic. The velocity potential far outside a fluid vortex of circulation Γ centered at the origin is $(\Gamma/2\pi)\theta(\boldsymbol{X})$. For a vortex state of the NLSE with winding number n, $2\Theta^0 = 2n\theta(\boldsymbol{X})$. The identification of $2\Theta^0$ with the velocity potential of the fluid vortex gives a determination of Γ, $\Gamma = 4\pi n$. The dynamics of widely separated fluid vortices has a simple characterization: A given vortex drifts with the current that would remain if that vortex were absent. Now consider the situation in asymptotic vortex dynamics of the NLSE: In (1.18), $2H$ is analogous to the velocity potential of the residual flow in the absence of a vortex at $\boldsymbol{X} = \boldsymbol{Q}$ and $2\boldsymbol{K}$ is analogous to the residual current at $\boldsymbol{Q}$. The dynamical boundary condition (1.20) states that a vortex at $\boldsymbol{Q}$ drifts with the residual current $2\boldsymbol{K}$.

1.5. Vortex dynamics of the NLHE

From the NLHE, it follows that the phase field Θ^0 satisfies the linear heat equation

$$\Theta^0_T = \Delta \Theta^0. \tag{1.23}$$

As before, asymptotic matching of core and far field expansions establishes the topological boundary condition (1.16) at each vortex position. Given the vortex trajectories and initial values of Θ^0 which satisfy the topological boundary conditions at the initial vortex positions, we may evolve Θ^0 forward in time. Effects of noninstantaneous communication between the vortices are clearly evident here. In principle, we can solve for the phase field Θ^0 as a functional of the initial conditions and vortex trajectories. Here, we choose to retain Θ^0 in the formulation of the asymptotic vortex dynamics. In this way, we naturally proceed to a genuine particle + field formulation of the asymptotic vortex dynamics.

1.6. Dynamical boundary condition

The formulation of the dynamical boundary condition is subtle because the far field and core expansions about a vortex position $\boldsymbol{X} = \boldsymbol{Q}$ have components which are logarithmic in the displacement from the vortex. The local behavior of the far field phase Θ^0 about $\boldsymbol{X} = \boldsymbol{Q}$ is given by

$$\Theta^0 = n\theta(\boldsymbol{R}) + \theta_0 + \tfrac{1}{2} n \log(R/\epsilon)\, J\dot{\boldsymbol{Q}} \cdot \boldsymbol{R} + \boldsymbol{K} \cdot \boldsymbol{R} + \mathcal{O}(R^2 \log R). \tag{1.24}$$

This expansion derives from the heat equation (1.23) and the topological boundary condition (1.16). $\theta_0 = \theta_0(T, \epsilon)$ and $\boldsymbol{K} = \boldsymbol{K}(T, \epsilon)$ are not determined by local analysis about $\boldsymbol{X} = \boldsymbol{Q}(T, \epsilon)$. They contain information about the initial conditions on Θ^0 and the past history of all the vortex trajectories: Their values are contained implicitly in the solution of the initial-boundary value problem for Θ^0.

The appearance of ϵ in the factor $\log(R/\epsilon)$ has a special significance: It renders the expansion (1.24) *scale covariant*. Each term of (1.24) is invariant under rescalings of the variables $\boldsymbol{X}, T, \boldsymbol{Q}$, induced by the change in the small but otherwise arbitrary gauge parameter ϵ. When we change ϵ to ϵ', we are effectively transforming X, T, Q into X', T', Q' defined by

$$X'/\epsilon' = X/\epsilon = x, \qquad T'/\epsilon'^2 = T/\epsilon^2 = t, \qquad Q'/\epsilon' = Q/\epsilon.$$

We easily deduce that $R/\epsilon \equiv [X - Q(T,\epsilon)]/\epsilon = [X' - Q'(T',\epsilon')]/\epsilon' \equiv R'/\epsilon$, $\theta(R) = \theta(R')$, and $JQ(T,\epsilon) \cdot R = J\dot{Q}'(T',\epsilon') \cdot R'$. Hence the local expansion in prime variables is

$$\Theta^0 = n\theta(R') + \theta_0' + \tfrac{1}{2} n \log(R'/\epsilon')\, J\dot{Q}' \cdot R' + K' \cdot R' + \mathcal{O}(R'^2 \log R').$$

Here, $\theta_0' = \theta_0'(T',\epsilon')$ denotes $\theta_0(T,\epsilon')$ expressed as a function of T', and $K' = K'(T',\epsilon')$ is related to $K = K(T,\epsilon)$ by the scaling $\epsilon K' = \epsilon K$.

As in the analysis of the NLSE, we think of the vector K as the "locally uniform component" of phase gradient about $X = Q$. It determines the vortex velocity: The dynamical boundary condition on Θ^0 at $X = Q$ is

$$m\dot{Q} = -2nJK + o(1), \tag{1.25}$$

where m is a constant determined from the structure of the vortex state (1.5).

1.7. Phenomenology

We begin with a simple exact solution of the free boundary problem for asymptotic vortex dynamics of the NLHE. There is a single isolated vortex, surrounded by a phase field Θ^0 whose gradient $\nabla\Theta^0$ converges to a constant vector K_∞ as $|X| \to \infty$. In this case, we anticipate that a uniform motion of the vortex with constant velocity ϵU is possible, and the problem is to compute U. Fig. 5 is a sketch of the situation, depicting the contours of constant phase Θ^0, and the direction of K_∞. The dashed line is a cut where Θ^0 has a necessary jump of 2π.

We may compute the far field phase exactly, and extract the asymptotic limit of this exact solution as $R \to 0$. The result modulo an additive constant is

$$\Theta^0 = n\theta(R) + \tfrac{1}{2} n \log(UR)\, JU \cdot R + K_\infty \cdot R + \mathcal{O}(R^2 \log R). \tag{1.26}$$

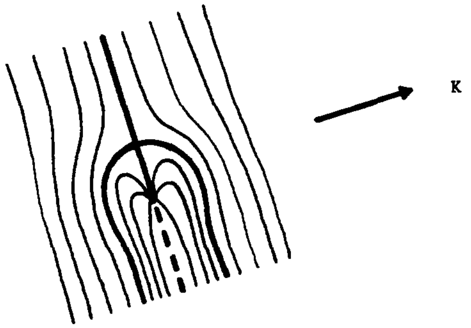

Fig. 5

(1.26) is equivalent to the general expression (1.24) if we take

$$K = \tfrac{1}{2} n \log(\epsilon U)\, JU + K_{\infty}. \tag{1.27}$$

The vortex velocity now follows from the dynamical boundary condition (1.25),

$$\log(\mathrm{e}^{-m}\epsilon U) U = 2nJK_{\infty} + o(1). \tag{1.28}$$

This velocity is orthogonal to K_{∞} with clockwise (counterclockwise) orientation for $n = +1$ ($n = -1$).

We present qualitative results about pairwise vortex interactions. Consider a pair of vortices at positions Q, Q' with winding numbers n and n', as depicted in fig. 6. An isolated vortex of winding number n' at position Q' has an associated phase gradient field $k(X) = n'J(X - Q')/|X - Q'|^2$. For the system of two vortices at positions Q, Q', $k(Q) = n'J\hat{R}/R$, $R \equiv Q - Q'$ provides a crude estimate of K, the locally uniform component of phase gradient at $X = Q$. The corresponding approximation to the vortex velocity $\dot{Q}$ is $\log(\mathrm{e}^{-m}\epsilon\dot{Q})\dot{Q} = 2nn'(\hat{R}/R)$. The approximate velocity of the vortex at Q' is given by $\log(\mathrm{e}^{-m}\epsilon\dot{Q}')\dot{Q}' = -2n'n(\hat{R}/R)$. Fig. 6 depicts these induced velocities for the cases $\operatorname{sgn} n' = \operatorname{sgn} n$,

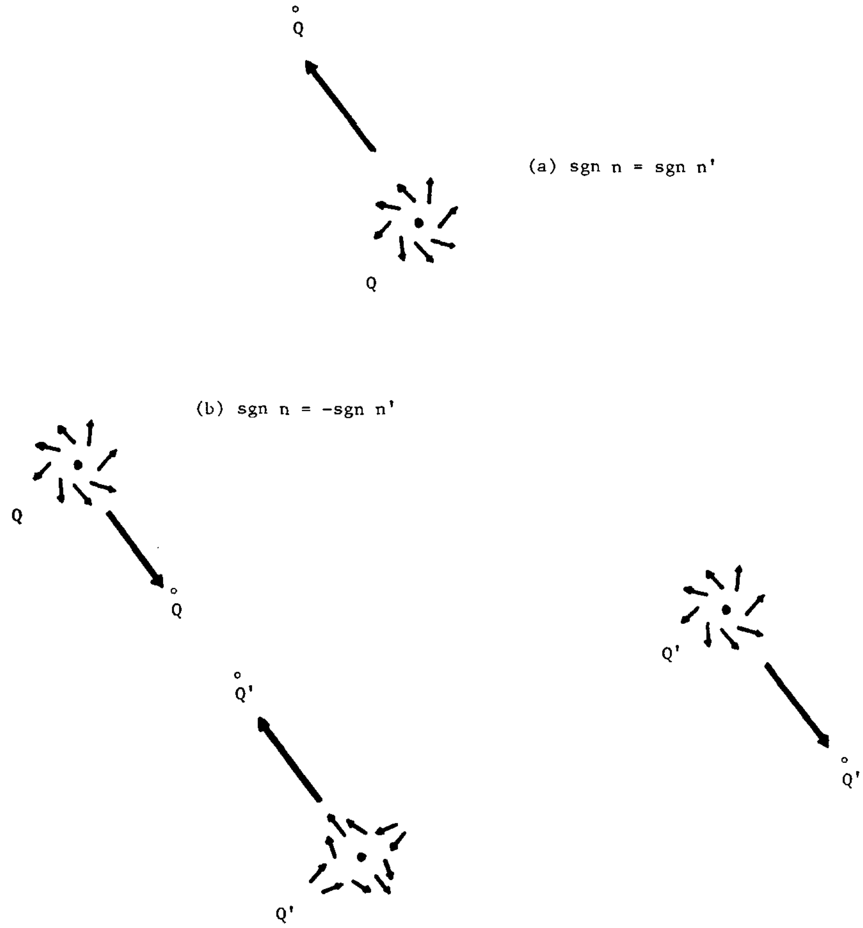

Fig. 6

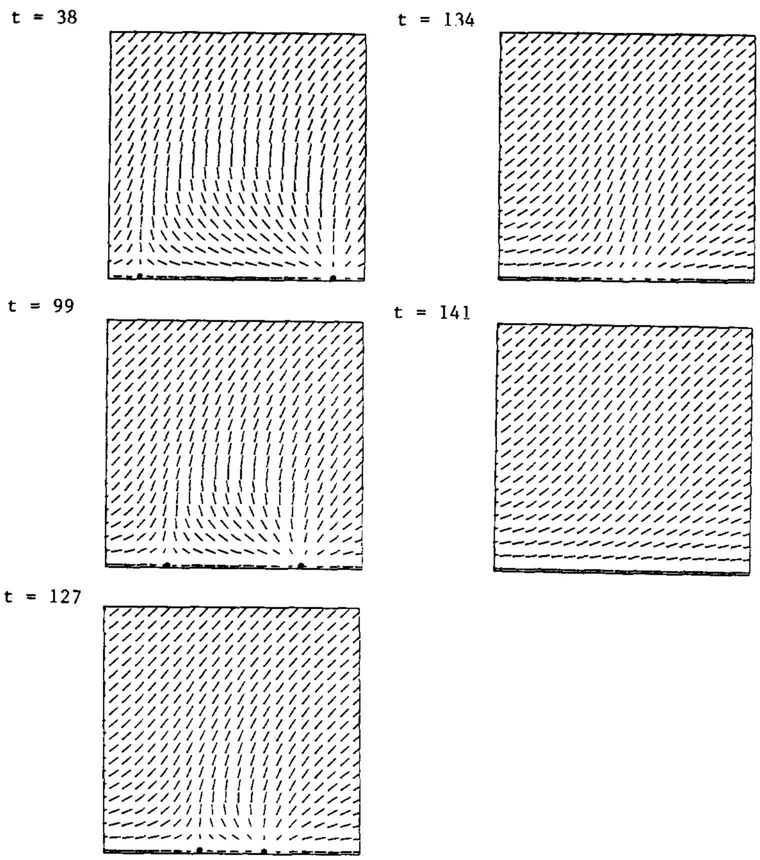

Fig. 7

sgn $n' = -\text{sgn}\, n$. We see that like (opposite) vortices repel (attract) each other with a force inversely proportional to their separation. The numerical movie of Carlson and Miller in fig. 3a documents the mutual repulsion of two $n = +1$ vortices produced by fission of an $n = 2$ vortex. Fig. 7 is another numerical movie, also by Carlson and Miller, documenting the attractive interaction between an $n = +1$ vortex and an $n = -1$ vortex. We observe that the final fate of these opposite vortices is merger and *mutual annihilation*.

2. Vortex dynamics of the nonlinear Schrödinger equation

We derive the equations governing the far field phase $\Theta^0(X, T, \epsilon)$ and the vortex trajectories $X = Q(T, \epsilon)$ in the limit $\epsilon \to 0$.

2.1. Far field expansion

We write the far field expansion in the polar form,

$$\psi \sim \Psi(X, T, \epsilon) = U(X, T, \epsilon) \exp[\mathrm{i}\Theta(X, T, \epsilon)]. \tag{2.1}$$

From the NLSE (1.1) it follows that U and Θ satisfy

$$(U^2-1)U = -\epsilon^2 U(\Theta_T + |\nabla\Theta|^2) + \epsilon^2 \Delta U, \tag{2.2a}$$
$$U\Delta\Theta = -U_T - 2\nabla U \cdot \nabla\Theta. \tag{2.2b}$$

It follows from (2.2a) that evolutions with U bounded away from zero have $U = 1 + \mathscr{O}(\epsilon^2)$. In this case, (2.2) becomes

$$U = 1 - \tfrac{1}{2}\epsilon^2(\Theta_T + |\nabla\Theta|^2) + \mathscr{O}(\epsilon^4), \tag{2.3a}$$
$$\Delta\Theta = \mathscr{O}(\epsilon^2). \tag{2.3b}$$

Given solutions for U, Θ we recover Ψ from

$$\Psi = U\,\mathrm{e}^{\mathrm{i}\Theta} \sim \left[1 - \tfrac{1}{2}\epsilon^2(\Theta_T + |\nabla\Theta|^2) + \mathscr{O}(\epsilon^4)\right]\mathrm{e}^{\mathrm{i}\Theta}. \tag{2.4}$$

The leading order far field expansion is

$$\Psi^0 = \mathrm{e}^{\mathrm{i}\Theta^0} \tag{2.5}$$

where Θ^0 satisfies Laplace's equation

$$\Delta\Theta^0 = 0. \tag{2.6}$$

2.2. Topological boundary condition and local structure of Ψ^0 about a vortex

Recall that the leading order core expansion ψ^0, valid for $\boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{Q}/\epsilon = \mathscr{O}(1)$, is the vortex state

$$\psi^0 = U(r)\exp[\mathrm{i}(n\theta + \theta_0)], \tag{2.7}$$

where r, θ are polar coordinates of $\boldsymbol{r}$, n is the winding number of the vortex at $\boldsymbol{X} = \boldsymbol{Q}$, and $\theta_0 = \theta_0(T, \epsilon)$ is a slowly varying phase shift. From (1.8) it follows that $U(r) = 1 + \mathscr{O}(1/r^2)$ as $r \to \infty$, so

$$\psi^0 = \exp[\mathrm{i}(n\theta + \theta_0)] + \mathscr{O}(1/r^2) \tag{2.8}$$

as $r \to \infty$. The obvious requirement for asymptotic matching of the leading order far field expansion Ψ^0 in (2.5) with the leading order core expansion in (2.7) is

$$\Theta^0 \sim n\theta(\boldsymbol{R}) + \theta_0 \tag{2.9}$$

as $\boldsymbol{R} \equiv \boldsymbol{X} - \boldsymbol{Q} = \epsilon\boldsymbol{r} \to 0$. (2.9) is the topological boundary condition on Θ^0 at $\boldsymbol{X} = \boldsymbol{Q}$.

We recall that the solution of Laplace's equation subject to the topological boundary condition takes the form

$$\Theta^0 = n\theta(\boldsymbol{R}) + H(\boldsymbol{X}, T, \epsilon), \tag{2.10}$$

where H is a harmonic function of X in a neighborhood of $X = Q$. The local approximations of Θ^0 and Ψ^0 about $X = Q$ are

$$\begin{aligned} &\Theta^0 \sim n\theta(\boldsymbol{R}) + \theta_0 + \boldsymbol{K}\cdot\boldsymbol{R} + \mathcal{O}(R^2), \\ &\Psi^0 \sim \exp\{\mathrm{i}[n\theta(\boldsymbol{R}) + \theta_0]\}\left[1 + \mathrm{i}\boldsymbol{K}\cdot\boldsymbol{R} + \mathcal{O}(R^2)\right], \end{aligned} \tag{2.11}$$

where $\theta_0 = \theta_0(T, \epsilon) \equiv H(\boldsymbol{Q}, T, \epsilon)$, and $\boldsymbol{K} = \boldsymbol{K}(T, \epsilon) \equiv \nabla H(\boldsymbol{Q}, T, \epsilon)$ is the locally uniform component of phase gradient at $X = Q$.

2.3. Core expansion

Substituting the core expansion

$$\psi \sim \psi(\boldsymbol{r}, T, \epsilon), \qquad \boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{Q}(T, \epsilon)/\epsilon$$

into the NLSE (1.1), we deduce

$$\Delta\psi + (1 - |\psi|^2)\psi = -\mathrm{i}(\epsilon^2\psi_T - \epsilon\dot{\boldsymbol{Q}}\cdot\nabla\psi), \tag{2.12}$$

where all spatial derivatives are with respect to $\boldsymbol{r}$. The two-term core expansion takes the form

$$\psi \sim \psi^0 + \epsilon\psi^1, \tag{2.13}$$

where ψ^0 is the vortex state (2.7) which satisfies the leading order perturbation equation

$$\Delta\psi^0 + (1 - |\psi^0|^2)\psi^0 = 0, \tag{2.14}$$

and ψ^1 satisfies the first-order perturbation equation

$$L\psi^1 = \mathrm{i}\dot{\boldsymbol{Q}}\cdot\nabla\psi^0. \tag{2.15}$$

L is the variational operator of the leading order equation (2.14) about ψ^0:

$$Lu \equiv \Delta u + (1 - 2|\psi^0|^2)u - \psi^{02}\bar{u}. \tag{2.16}$$

We impose a higher-order matching, between the leading order far field expansion Ψ^0 and the two-term core expansion $\psi^0 + \epsilon\psi^1$. From the results for ψ^0 and Ψ^0 in (2.8), (2.11), we deduce

$$\begin{aligned} &\psi^0(\boldsymbol{r}, T, \epsilon) + \epsilon\psi^1(\boldsymbol{r}, T, \epsilon) - \Psi^0(\epsilon\boldsymbol{r}, T, \epsilon) \\ &\quad = \epsilon\psi^1(\boldsymbol{r}, T, \epsilon) - \mathrm{i}\epsilon\boldsymbol{K}\cdot\boldsymbol{r}\exp[\mathrm{i}(n\theta + \theta_0)] + \mathcal{O}(\epsilon^2 r^2 + 1/r^2). \end{aligned} \tag{2.17}$$

Asymptotic matching requires that $\epsilon\boldsymbol{K}\cdot\boldsymbol{r} = \mathcal{O}(\epsilon r)$ be larger in magnitude than the error terms $\mathcal{O}(\epsilon^2 r^2)$ and $\mathcal{O}(1/r^2)$, and that the difference between ψ^1 and $\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{r}$ be smaller in magnitude than $\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{r} = \mathcal{O}(r)$. Hence, the matching condition is

$$\psi^1(\boldsymbol{r}, T) - \mathrm{i}\boldsymbol{K}\cdot\boldsymbol{r}\exp[\mathrm{i}(n\theta + \theta_0)] = o(r), \tag{2.18}$$

in the overlap domain

$$r = \mathcal{O}(\epsilon^p), \qquad -1 < p < -\tfrac{1}{3}.$$

It is simple to check that this matching condition is formally consistent with the perturbation equation (2.15) for ψ^1.

2.4. The dynamical boundary condition

The perturbation equation (2.15) for ψ^1 subject to the matching condition (2.18) determines the vortex velocity $\dot{\boldsymbol{Q}}$ as a function of $\boldsymbol{K}$ in the limit $\epsilon \to 0$. We present this determination here. The variational operator L in (2.16) is self-adjoint: Given smooth functions $u(\boldsymbol{x})$, $v(\boldsymbol{x})$ defined for $\boldsymbol{x} \in \mathbb{R}^2$, and D any region of $\mathbb{R}^2$,

$$\int_{\mathrm{D}} \mathrm{Re}\left(u\overline{Lv} - v\overline{Lu}\right) \mathrm{d}\boldsymbol{x} = \int_{\partial \mathrm{D}} \mathrm{Re}\left(u\,\overline{\partial_n v} - v\,\overline{\partial_n u}\right) \mathrm{d}l. \tag{2.19}$$

Here, ∂_n denotes outward normal derivative on ∂D, and dl is the arclength element along ∂D. We apply this formula with $u = \hat{\boldsymbol{e}} \cdot \nabla\psi^0$, where $\hat{\boldsymbol{e}}$ is any fixed unit vector, and $v = \psi^1$. We take D to be a disk of radius r_0 about $r = 0$. We impose $r_0 = \mathcal{O}(\epsilon^p)$, $-1 < p < -\frac{1}{3}$, so that the circle $|\boldsymbol{r}| = r_0$ lies in the overlap domain where the matching condition (2.18) applies. Taking the directional derivative $\hat{\boldsymbol{e}} \cdot \nabla$ of the leading order perturbation equation (2.14), we deduce $Lu = L(\hat{\boldsymbol{e}} \cdot \nabla\psi^0) = 0$. From the first-order perturbation equation (2.15) we have $Lv = L\psi^1 = \mathrm{i}\dot{\boldsymbol{Q}} \cdot \nabla\psi^0$. Hence, (2.19) becomes

$$\int_{|\boldsymbol{r}|<r_0} \mathrm{Re}\left[\mathrm{i}(\hat{\boldsymbol{e}} \cdot \nabla\psi^0)(\dot{\boldsymbol{Q}} \cdot \nabla\overline{\psi_0})\right] \mathrm{d}\boldsymbol{r} = \int_{|\boldsymbol{r}|=r_0} \mathrm{Re}\left[(\hat{\boldsymbol{e}} \cdot \nabla\psi^0)\,\partial_r\overline{\psi^1} - \psi^1\partial_r\left(\hat{\boldsymbol{e}} \cdot \nabla\overline{\psi^0}\right)\right] \mathrm{d}l. \tag{2.20}$$

Here, ψ^0 is the known vortex state (2.7), and the asymptotic behavior of ψ^1 on $|\boldsymbol{r}| = r_0$ is given by the matching condition (2.18). Hence, we may explicitly evaluate the limit of (2.20) as $\epsilon \to 0$. We obtain the equation

$$(\dot{\boldsymbol{Q}} - 2\boldsymbol{K}) \cdot J\hat{\boldsymbol{e}} = o(1). \tag{2.21}$$

Since (2.21) holds for any choice of the unit vector $\hat{\boldsymbol{e}}$, it follows that

$$\dot{\boldsymbol{Q}} = 2\boldsymbol{K} + o(1). \tag{2.22}$$

3. Vortex dynamics of the nonlinear heat equation

The derivation of an asymptotic vortex dynamics for the NLHE follows in broad outline the treatment of the NLSE. But there is an additional degree of difficulty – the core and far field expansions both have components which are logarithmic in the displacement from a vortex center. A direct manifestation of these logarithmic terms is the appearance of $\log \epsilon$ as a parameter in asymptotic expansions of ψ. Recall that a logarithmic dependence upon ϵ is "slower" than any power of ϵ, and we proceed by balancing $\mathcal{O}(\epsilon^m)$ and $\mathcal{O}(\epsilon^m \log \epsilon)$ terms in the same perturbation equation. In particular, the leading order far field phase $\Theta^0(X, T, \epsilon)$ has a logarithmic dependence upon ϵ, and the dynamical boundary conditions at vortex positions leads to scaled vortex velocities $\dot{Q}$ which are $\mathscr{O}(1/|\log \epsilon|)$.

3.1. The far field

We write the far field expansion in polar form, as in (2.1). Substituting (2.1) into the NLHE leads to the pair of equations

$$(U^2-1)U = -\epsilon^2 U_T + \epsilon^2(\Delta U - |\nabla\Theta|^2 U), \tag{3.1a}$$

$$U\Theta_T = U\Delta\Theta + 2\nabla U \cdot \nabla\Theta. \tag{3.1b}$$

It follows from (3.1a) that solutions with U bounded away from zero have $U = 1 + \mathscr{O}(\epsilon^2)$, in which case (3.1) reduces to

$$U = 1 - \tfrac{1}{2}\epsilon^2|\nabla\Theta|^2 + \mathscr{O}(\epsilon^4), \tag{3.2a}$$

$$\Theta_T - \Delta\Theta = \mathscr{O}(\epsilon^2). \tag{3.2b}$$

The leading order approximation Θ^0 of the far field phase Θ satisfies the heat equation

$$\Theta^0_T = \Delta\Theta^0. \tag{3.3}$$

The leading order far field approximation Ψ^0 of ψ is related to Θ^0 by

$$\Psi^0 = \mathrm{e}^{\mathrm{i}\Theta^0}. \tag{3.4}$$

3.2. The local structure of Ψ^0 about a vortex

Θ^0 satisfies the topological boundary condition (2.9) at vortex positions $\boldsymbol{X} = \boldsymbol{Q}$. Given an initial condition on Θ^0 at $T = 0$ which satisfies the topological boundary condition at the initial vortex positions $\boldsymbol{Q}(0, \epsilon)$, and given the vortex trajectories $\boldsymbol{Q}(T, \epsilon)$ in $T > 0$, we may determine Θ^0 in $T > 0$ which satisfies the heat equation (3.4) subject to the topological boundary condition at each vortex position. The remaining problem is to determine the dynamics of the vortices in response to the evolution of Θ^0. The topological boundary condition (2.9) already provides the leading term of this expansion. We can systematically determine the forms of higher corrections. To this end, it is convenient to write the local expansion of Θ^0 in terms of the translating spatial variable $\boldsymbol{R} \equiv \boldsymbol{X} - \boldsymbol{Q}$. The heat equation (3.3) in $\boldsymbol{R}, T$ coordinates reads

$$\Delta\Theta^0 = -\dot{\boldsymbol{Q}} \cdot \nabla\Theta^0 + \Theta^0_T. \tag{3.5}$$

All spatial derivatives are with respect to $\boldsymbol{R}$. We write the local expansion of Θ^0 about $R = 0$ in the form

$$\Theta^0 = n\theta(\boldsymbol{R}) + \theta_0(T, \epsilon) + H(\boldsymbol{R}, T, \epsilon). \tag{3.6}$$

Consistency of (3.6) with the topological boundary condition requires H to be continuous in a neighborhood of $R = 0$, with $H(0, T, \epsilon) = 0$. The two-term asymptotic expansion of H as $R \to 0$ takes the form

$$H = (\log R)\boldsymbol{A}\cdot\boldsymbol{R} + \boldsymbol{K}\cdot\boldsymbol{R} + \mathscr{O}(R^2 \log R), \tag{3.7}$$

where $\boldsymbol{A}$ and $\boldsymbol{K}$ are vectors which are functions of T but independent of $\boldsymbol{R}$. Substituting (3.6) into (3.5) we obtain a determination of $\boldsymbol{A}$, $\boldsymbol{A} = \frac{1}{2}nJ\dot{\boldsymbol{Q}}$, but no determination of $\boldsymbol{K}$. In summary, the local expansion of Θ^0 about $R = 0$ takes the form

$$\Theta^0 = n\theta(\boldsymbol{R}) + \theta_0 + \tfrac{1}{2}n(\log R)J\dot{\boldsymbol{Q}}\cdot\boldsymbol{R} + \boldsymbol{K}\cdot\boldsymbol{R} + \mathscr{O}(R^2 \log R), \tag{3.8}$$

where $\boldsymbol{K}$ is not determined by the local analysis.

For any positive constant a, we may redefine $\boldsymbol{K}$ so that the expansion

$$\Theta^0 = n\theta(\boldsymbol{R}) + \theta_0 + \tfrac{1}{2}n\log(aR)\,J\dot{\boldsymbol{Q}}\cdot\boldsymbol{R} + \boldsymbol{K}\cdot\boldsymbol{R} + \mathscr{O}(R^2 \log R) \tag{3.9}$$

is equivalent to (3.8). The choice $a = 1/\epsilon$ leads to a *scale covariant* form of the expansion,

$$\Theta^0 = n\theta(\boldsymbol{R}) + \theta_0 + \tfrac{1}{2}n\log(R/\epsilon)\,J\dot{\boldsymbol{Q}}\cdot\boldsymbol{R} + \boldsymbol{K}\cdot\boldsymbol{R} + \mathscr{O}(R^2 \log R). \tag{3.10}$$

Recall from the discussion of section 1 that each term is invariant under rescalings of the variables $\boldsymbol{R}, T, \boldsymbol{Q}$ induced by a change in the small but otherwise arbitrary gauge parameter ϵ. We work with the scale covariant expansion (3.10) of Θ^0. The corresponding expansion of $\Psi^0 = \mathrm{e}^{\mathrm{i}\Theta^0}$ is

$$\Psi^0 = \exp\{\mathrm{i}[n\theta(\boldsymbol{R}) + \theta_0]\}\left[1 + \tfrac{1}{2}\mathrm{i}n\log(R/\epsilon)\,J\dot{\boldsymbol{Q}}\cdot\boldsymbol{R} + \mathrm{i}\boldsymbol{K}\cdot\boldsymbol{R} + \mathscr{O}(R^2 \log R)\right]. \tag{3.11}$$

As noted in section 1, the evolution of $\boldsymbol{K}$ in time T is contained implicitly in the solution of the initial-boundary value problem for Θ^0. We call $\boldsymbol{K}$ the "locally uniform component of phase gradient at $\boldsymbol{X} = \boldsymbol{Q}$".

3.3. Core expansion

The NLHE (1.2) in the "core" variables $\boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{Q}/\epsilon$, $T \equiv \epsilon^2 t$ reads

$$\Delta\psi + (1 - |\psi|^2)\psi = \epsilon^2\psi_T - \epsilon\dot{\boldsymbol{Q}}\cdot\nabla\psi. \tag{3.12}$$

The two-term core expansion takes the form

$$\psi \sim \psi^0 + \epsilon\psi^1. \tag{3.13}$$

Here, ψ^0 is the vortex state (2.7). ψ^0 satisfies the leading order perturbation equation (2.14) as in the analysis of the NLSE, and the first-order perturbation equation is

$$L\psi^1 = -\dot{\boldsymbol{Q}}\cdot\nabla\psi^0, \tag{3.14}$$

where L is the variational operator (2.16).

As before, we impose an asymptotic matching of the leading order far field expansion Ψ^0 and the two-term core expansion $\psi^0 + \epsilon\psi^1$. From (2.8), (3.11) it follows that

$$\psi^0(\boldsymbol{r},T,\epsilon) + \psi^1(\boldsymbol{r},T,\epsilon) - \Psi^0(\epsilon\boldsymbol{r},T,\epsilon)$$
$$= \epsilon\psi^1(\boldsymbol{r},T,\epsilon) - \mathrm{i}\epsilon\left[\tfrac{1}{2}n(\log r)\,J\dot{\boldsymbol{Q}} + \boldsymbol{K}\right]\cdot\boldsymbol{r}\exp[\mathrm{i}(n\theta+\theta_0)] + \mathscr{O}\left(\epsilon^2 r^2\log\epsilon r + \frac{1}{r^2}\right). \tag{3.15}$$

The matching condition is that

$$\psi^1(\boldsymbol{r},T,\epsilon) = \mathrm{i}\left[\tfrac{1}{2}n(\log r)\,J\dot{\boldsymbol{Q}} + \boldsymbol{K}\right]\cdot\boldsymbol{r} + o(r) \tag{3.16}$$

in the overlap domain $r = \mathscr{O}(\epsilon^p)$, $-1 < p < -\frac{1}{3}$. The matching condition (3.16) is consistent with formally constructed asymptotic solutions to the first-order perturbation equation (3.14).

3.4. The dynamical boundary condition

The first-order perturbation equation (3.14) for ψ^1 subject to the effective boundary condition (3.15) determines the leading approximation to the vortex velocity $\boldsymbol{K}$ in the limit $\epsilon \to 0$. From (3.14), (3.15) we derive the identity

$$\int_{|\boldsymbol{r}|<r_0} \mathrm{Re}\left[-(\hat{\boldsymbol{e}}\cdot\nabla\psi^0)(\dot{\boldsymbol{Q}}\cdot\nabla\overline{\psi^0})\right]\mathrm{d}\boldsymbol{r} = \int_{|\boldsymbol{r}|=r_0} \mathrm{Re}\left[(\hat{\boldsymbol{e}}\cdot\nabla\psi^0)\,\partial_r\overline{\psi^1} - \psi^1\partial_r(\hat{\boldsymbol{e}}\cdot\nabla\overline{\psi^0})\right]\mathrm{d}l, \tag{3.17}$$

which is the direct counterpart of (2.20). Again the procedure is to evaluate both sides of (3.17) with r_0 in the overlap domain, $r_0 = \mathscr{O}(\epsilon^{-p})$, $-1 < p < -\frac{1}{3}$ and take the limit of (3.17) as $\epsilon \to 0$. The actual mechanics of this calculation has some interesting features. We observe the cancellation of components which are logarithmic in r_0, and the balance of $\mathscr{O}(1)$ components independent of r_0 gives the asymptotic determination of $\dot{\boldsymbol{Q}}$ in terms of $\boldsymbol{K}$.

Substituting into the l.h.s. of (3.17) the representation $\psi^0 = U(r)\exp\{\mathrm{i}[n\theta + \theta_0(T)]\}$ of the vortex state, we find that

$$\text{l.h.s.} = -\pi(\hat{\boldsymbol{e}}\cdot\dot{\boldsymbol{Q}})\int_0^{r_0}\left(U_r^2 + \frac{U^2}{r^2}\right)r\,\mathrm{d}r. \tag{3.18}$$

Here, we used $n = +1$ or -1 so $n^2 = 1$. From the asymptotic behavior of $U(r)$ as $r \to \infty$, $U(r) = 1 + \mathscr{O}(1/r^2)$, it follows that

$$\text{l.h.s.} = -\pi(\log r_0 + \alpha)(\hat{\boldsymbol{e}}\cdot\dot{\boldsymbol{Q}}) + \mathscr{O}(1/r_0^2) \tag{3.19}$$

as $r_0 \to \infty$. Here, α is a constant independent of r_0,

$$\alpha \equiv \lim_{r_0\to\infty}\left[\int_0^{r_0}\left(U_r^2 + \frac{U^2}{r^2}\right)r\,\mathrm{d}r - \log r_0\right]. \tag{3.20}$$

Substituting into the r.h.s. of (3.17) the asymptotic form (3.16) of ψ^1 valid in the overlap domain $r = \mathcal{O}(\epsilon^p)$, $-1 < p < -\frac{1}{3}$, we find that

$$\text{r.h.s.} = -\pi(\log r_0 + 1)(\hat{e}\cdot\dot{Q}) + 2\pi n\hat{e}\cdot JK + o(1). \tag{3.21}$$

In imposing equality of the l.h.s. and r.h.s. as given by (3.19), (3.21) we see that the $\log r_0$ terms automatically cancel, and that the remaining terms lead to the condition

$$(m\dot{Q} + 2nJK)\cdot\hat{e} = o(1) \tag{3.22}$$

as $\epsilon \to 0$. Here, $m \equiv 1 - \alpha$. Since (3.22) holds for all unit vectors $\hat{e}$, it follows that

$$m\dot{Q} = -2nJK + o(1)$$

as $\epsilon \to 0$. This relationship between the vortex velocity $\dot{Q}$ and the locally uniform component of phase gradient K at $X = Q$ is the dynamical boundary condition on Θ^0 at $X = Q$.

3.5. Vortex in a uniform phase gradient

In the discussion of phenomenology in section 1, we examined the motion of a single isolated vortex surrounded by a phase field Θ^0 whose gradient asymptotes to a constant vector K_∞ at spatial infinity. The formula (1.28) for the velocity is based upon the relationship (1.27) between the phase gradient K_∞ at spatial infinity and K, the locally uniform component of phase gradient at the vortex. This result is based upon an exact solution for Θ^0. The implementation of the topological boundary condition in the construction of this solution is of particular interest.

Let (X, Y) be a right-handed Cartesian coordinate system of $\mathbb{R}^2$ centered on the uniformly moving vortex center. The X axis is parallel to K_∞ and we anticipate that the vortex velocity U is orthogonal to K_∞, so $U = U\hat{Y}$. Solutions of the heat equation (3.3) which are time independent in the translating (X, Y) frame satisfy

$$\Delta\Theta^0 + U\Theta^0_Y = 0. \tag{3.23}$$

We require a solution which satisfies the topological boundary condition,

$$\Theta^0 \sim n\theta(R) \quad \text{as } R \equiv |X| \to \infty, \tag{3.24a}$$

and

$$\nabla\Theta^0 \to K_\infty \quad \text{as } R \to \infty. \tag{3.24b}$$

The topological boundary condition requires Θ^0 to be multivalued. We compute a branch of Θ^0 in the slit plane with the negative Y axis excluded: The values of $\theta(R)$ are taken in the interval $-\pi/2 < \theta < 3\pi/2$. The jump in Θ^0 across the slit is $2\pi n$.

The solution of the elliptic boundary value problem (3.23)–(3.24) may be constructed from the *fundamental solution* $G(\boldsymbol{x})$, satisfying

$$\Delta G + UG_Y = 2\pi n\delta(\boldsymbol{X}), \tag{3.25a}$$

$$G \to 0 \quad \text{as } R \to \infty. \tag{3.25b}$$

The process is quite analogous to determining $\theta(X) = \arctan(Y/X)$ from its harmonic conjugate $\log[(X^2+Y^2)^{1/2}]$, which is proportional to the fundamental solution of the Laplacian in two dimensions.

These are the details: From the divergence theorem and (3.25) we deduce

$$\int_C -(G_Y + UG)\,\mathrm{d}X + G_X\,\mathrm{d}Y = 2\pi n, \tag{3.26}$$

where C is a counterclockwise loop enclosing $\boldsymbol{X} = 0$. Comparing (3.26) to the topological constraint

$$\int_C \nabla\Theta^0 \cdot \mathrm{d}\boldsymbol{X} = \int_C \Theta^0_X\,\mathrm{d}X + \Theta^0_Y\,\mathrm{d}Y = 2\pi n,$$

we see that the latter is automatically satisfied if Θ^0 is related to G in $|\boldsymbol{X}| > 0$ by

$$\Theta^0_X = -G_Y - UG, \tag{3.27a}$$

$$\Theta^0_Y = G_X. \tag{3.27b}$$

These relations may be thought of as analogs of Cauchy–Riemann equations with Θ^0 and G "conjugate" to each other. We check that (3.27) is consistent with (3.23) and (3.25a). In $|\boldsymbol{X}| > 0$, it follows from (3.27) that

$$\Theta^0_{XX} + \Theta^0_{YY} + U\Theta^0_Y = -G_{YX} - UG_X + G_{XY} + UG_X = 0,$$

and

$$G_{XX} + G_{YY} + UG_Y = \Theta^0_{YX} - \Theta^0_{XY} = 0.$$

A consequence of (3.27) is that solutions Θ^0 of (3.23) satisfying the topological boundary condition (3.24a) may be computed from a line integral,

$$\Theta^0 = \int_\gamma \Theta^0_X\,\mathrm{d}X + \Theta^0_Y\,\mathrm{d}Y = \int_\gamma -(G_Y + UG)\,\mathrm{d}X + G_X\,\mathrm{d}Y$$

$$= \int_\gamma (\nabla G + G\hat{Y}) \cdot \hat{\boldsymbol{n}}\,\mathrm{d}l. \tag{3.28}$$

Here, γ is any path in the slit plane from a fixed point $\boldsymbol{X}_0$ to $\boldsymbol{X}$, $\hat{\boldsymbol{n}}$ is the unit normal to γ and l is the archlength along γ.

The fundamental solution G satisfying (3.25) is

$$G(R,\theta) = -n\exp(-\tfrac{1}{2}UR\sin\theta)K_0(\tfrac{1}{2}|U|R), \tag{3.29}$$

where R, θ are polar coordinates of $\boldsymbol{X}$, and K_0 is the Bessel function of order zero. From this result for G, we see that $G_X = 0$ along the negative Y axis, which is the slit in the X, Y plane. It follows from (3.28) that Θ^0 is constant along either side of the cut. Given $\boldsymbol{X}$, we take the path γ in (3.28) to be the arc of a circle of radius $|\boldsymbol{X}|$ extending from the cut to $\boldsymbol{X}$ in the counterclockwise direction. The resulting solution for Θ^0, up to a constant of integration, is

$$\Theta^0(R,\theta) = nR\int_{-\pi/2}^{\theta}[G_R(R,\zeta) + U\sin\zeta\, G(R,\zeta)]\,\mathrm{d}\zeta. \tag{3.30}$$

It is simple to show that Θ^0 given in (3.30) satisfies $\nabla\Theta^0 \to 0$ as $R \to \infty$. To obtain a solution which satisfies the boundary condition $\nabla\Theta^0 \to \boldsymbol{K}_\infty$ as $R \to \infty$, we add the term $K_\infty X = K_\infty R\cos\theta$ to (3.30), which is a solution of (3.23) by itself. Hence, the exact solution of the boundary value problem (3.23), (3.24) to within a constant is

$$\Theta^0(R,\theta) = nR\int_{-\pi/2}^{\theta}[G_R(R,\zeta) + U\sin\zeta\, G(R,\zeta)]\,\mathrm{d}\zeta + K_\infty R\cos\theta. \tag{3.31}$$

The asymptotic expansion of this exact solution as $R \to 0$ is

$$\begin{aligned}\Theta^0 &= n\theta(\boldsymbol{R}) - [\tfrac{1}{2}nU\log(|U|R) - K_\infty]R\cos\theta + \mathscr{O}(R^2\log R)\\ &= n\theta(\boldsymbol{R}) + \tfrac{1}{2}n\log(|U|R)\boldsymbol{J}\boldsymbol{U}\cdot\boldsymbol{R} + \boldsymbol{K}_\infty\cdot\boldsymbol{R} + \mathscr{O}(R^2\log R).\end{aligned} \tag{3.32}$$

Comparing this result for Θ^0 with the general expansion (3.10) about a vortex with velocity $\dot{\boldsymbol{Q}} = \boldsymbol{U} = U\hat{\boldsymbol{y}}$, we deduce the relationship between $\boldsymbol{K}_\infty$ and $\boldsymbol{K}$, and the locally uniform component of the phase gradient at the vortex,

$$\boldsymbol{K} = \tfrac{1}{2}n\log(\epsilon|U|)\boldsymbol{J}\boldsymbol{U} + \boldsymbol{K}_\infty. \tag{3.33}$$

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