Solution To PHYS622 Homework Set 9

1 Solution1

$$V_{\pm}^{(1)} = \mp\frac{V_x \pm iV_y}{\sqrt{2}}, V_0^{(1)} = V_z$$

$$V_{+}^{\prime(1)} = -\frac{1}{\sqrt{2}}[(V_x\cos\beta + V_z\sin\beta) + iV_y]$$

$$V_{0}^{\prime(1)} = -V_x\sin\beta + V_z\cos\beta$$

$$V_{-1}^{\prime(1)} = \frac{1}{\sqrt{2}}[(V_x\cos\beta + V_z\sin\beta) - iV_y]$$

If we rotate a vector $\vec{V}$ about the y-axis by an angle β,the new vector is:

$$\begin{bmatrix} V_x' \\ V_y' \\ V_z' \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

2 Solution2

(a).

$$T_{+1}^{(1)} = \frac{i}{2}[(U_yV_z - U_zV_y) + i(U_zV_x - V_zU_x)]$$

$$T_{0}^{(1)} = \frac{i}{\sqrt{2}i}[(U_xV_y - V_xU_y)]$$

$$T_{-1}^{(1)} = \frac{i}{2i}[(U_yV_z - U_zV_y) - i(U_zV_x - V_zU_x)]$$

(b).

$$T_{+2}^{(2)} = \frac{1}{2}[(U_xV_x - U_yV_y) + i(U_xV_y + V_xU_y)]$$

$$T_{-2}^{(2)} = \frac{1}{2}[(U_xV_x - U_yV_y) - i(U_xV_y + V_xU_y)]$$

$$T_{+1}^{(2)} = -\frac{1}{2}[(U_xV_z + U_zV_x) + i(U_yV_z + V_yU_z)]$$

$$T_{-1}^{(2)} = -\frac{1}{2}[(U_xV_z + U_zV_x) - i(U_yV_z + V_yU_z)]$$

$$T_{0}^{(2)} = \frac{1}{\sqrt{6}}[(2U_zV_z - U_xV_x - U_yV_y)]$$

3 Solution3

$$A = \frac{l + 1 + \vec{l}\cdot\vec{\sigma}}{2l+1}, B = \frac{l - \vec{l}\cdot\vec{\sigma}}{2l+1}$$

Note $\vec{l}\cdot\vec{\sigma} = \vec{J}^2 - \vec{l}^2 - \vec{S}^2$

$$A|j = l + 1/2, m >= \frac{l + 1 + \vec{l}\cdot\vec{\sigma}}{2l + 1}|j = l + 1/2, m >= \frac{l + 1 + \vec{J}^2 - \vec{l}^2 - \vec{S}^2}{2l + 1}|j = l + 1/2, m >=$$
$$\frac{l + 1 + (l + 1/2)(l + 3/2) - l(l + 1) - 1/2(1/2 + 1)}{2l + 1}|j = l + 1/2, m >= |j = l + 1/2, m >$$
$$A|j = l - 1/2, m >= 0$$

Similarly

$$B|j = l + 1/2, m >= 0$$
$$B|j = l - 1/2, m >= |j = l - 1/2, m >$$

Write an arbitrary state in terms of the angular momentum states,

$$\psi = \sum_m C_m|j = l + 1/2, m > + \sum_m D_m|j = l - 1/2, m >$$
$$A\psi = \sum_m C_m|j = l + 1/2, m >$$
$$A^2\Psi = A\sum_m C_m|j = l + 1/2, m >= \sum_m C_m|j = l + 1/2, m >$$
$$B\psi = \sum_m D_m|j = l - 1/2, m >$$
$$B^2\Psi = B\sum_m C_m|j = l - 1/2, m >= \sum_m C_m|j = l - 1/2, m >$$

Hence, $A^2 = A$, $B^2 = B$.

$$< m - 1/2, 1/2|l + 1/2, m >= \sqrt{\frac{l + m + 1/2}{2l + 1}}$$
$$< m - 1/2, 1/2|l - 1/2, m >= \sqrt{\frac{l - m + 1/2}{2l + 1}}$$
$$< m + 1/2, -1/2|l + 1/2, m >= \sqrt{\frac{l - m + 1/2}{2l + 1}}$$
$$< m + 1/2, -1/2|l - 1/2, m >= \sqrt{\frac{l + m + 1/2}{2l + 1}}$$

4 Solution4

(a).

$$< n', l', m'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|n, l, m >=< l, 1; m, \pm 1|l, 1; l', m' > \frac{< n'l'||T^{(1)}||nl >}{\sqrt{2l + 1}}$$

$$< n'l'm'|z|nlm > = < l,1;m,0|l,1;l',m' > \frac{< n'l'||T^{(1)}||nl >}{\sqrt{2l+1}}$$

$$\Longrightarrow \frac{< n',l',m'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|n,l,m >}{< n'l'm'|z|nlm >} = \frac{< l,1;m,\pm 1|l,1;l',m' >}{< l,1;m,0|l,1;l',m' >}$$

(b).

$$< n'l'm'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|nlm > = \int\int\int R^*_{n'l'}(r) Y^{*m'}_{l'}(\theta,\phi)(\mp\frac{1}{\sqrt{2}})(x \pm iy) R_{nl}(r) Y^m_l(\theta,\phi) r^2 dr \sin\theta d\theta d\phi$$

$$< n'l'm'|z|nlm > = \int\int\int R^*_{n'l'}(r) Y^{*m'}_{l'}(\theta,\phi)(r\cos\theta) R_{nl}(r) Y^m_l(\theta,\phi) r^2 dr \sin\theta d\theta d\phi$$

$$\Longrightarrow \frac{< n'l'm'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|nlm >}{< n'l'm'|z|nlm >} = \frac{\mp\frac{1}{\sqrt{2}} \int\int Y^{*m'}_{l'}(\theta,\phi) Y^m_l(\theta,\phi) e^{\pm i\phi} \sin d\theta d\Omega}{\int\int Y^{*m'}_{l'}(\theta,\phi) Y^m_l(\theta,\phi)\cos\theta d\Omega}$$

Calculate the integration in the numerator and denominator,we have

$$N = \sqrt{\frac{2l+1}{2l'+1}} < l,1;0,0|l,1;l',0 >< l,1;m \pm 1|l,1;l',m' >$$

$$D = \sqrt{\frac{2l+1}{2l'+1}} < l,1;0,0|l,1;l',0 >< l,1;m,0|l,1;l',m' >$$

$$\frac{< n'l'm'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|nlm >}{< n'l'm'|z|nlm >} = \frac{< l,1;0,0|l,1;l',0 >< l,1;m \pm 1|l,1;l',m' >}{< l,1;0,0|l,1;l',0 >< l,1;m,0|l,1;l',m' >}$$

5 Solution5

(a).

$$xy = \frac{T^{(2)}_{+2} - T^{(2)}_{-2}}{2i}$$

$$xz = \frac{T^{(2)}_{-1} - T^{(2)}_{+1}}{2}$$

$$x^2 - y^2 = T^{(2)}_{+2} + T^{(2)}_{-2}$$

(b).

$$e < \alpha,j,m'|(x^2 - y^2)|\alpha,j,m = j > = \frac{Q}{\sqrt{6}} \frac{< j,2;m = j,q = -2|j,2;j,m' >}{< j,2;m = j,q = 0|j,2;j,m = j >}$$

6 Solution6

$$H_{int} = A(3S_z^2 - S^2) + B(S_+^2 + S_-^2)$$

$$A = \frac{eQ}{4s(s-1)\hbar^2}\left(\frac{\partial^2\phi}{\partial z^2}\right)_0$$
$$B = \frac{eQ}{4s(s-1)\hbar^2}\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \frac{1}{2}\left(\frac{\partial^2\phi}{\partial z^2}\right)_0\right]$$

Write H_{int} as matrix form

$$H_{int} = \hbar^2 \begin{bmatrix} 3A & 0 & 2\sqrt{2} & 0 \\ 0 & -3A & 0 & 2\sqrt{3}B \\ 2\sqrt{3}B & 0 & -3A & 0 \\ 0 & 2\sqrt{3}B & 0 & 3A \end{bmatrix}$$

The eigenvalues are

$$E_{1,2} = -\sqrt{3}\sqrt{3A^2+4B^2}\hbar^2$$
$$E_{3,4} = \sqrt{3}\sqrt{3A^2+4B^2}\hbar^2$$

Two-fold degeneracy. Eigenvectors are

$$\begin{pmatrix} 0 \\ \frac{-3A+\sqrt{3}\sqrt{3A^2+4B^2}}{2\sqrt{3}B} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{-3A+\sqrt{3}\sqrt{3A^2+4B^2}}{2\sqrt{3}B} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{-3A+\sqrt{3}\sqrt{3A^2+4B^2}}{2\sqrt{3}B} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{3A+\sqrt{3}\sqrt{3A^2+4B^2}}{2\sqrt{3}B} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

7 Solution7

$$\vec{\mu} = \vec{\mu}_l + \vec{\mu}_s = \frac{e}{2mc}(\vec{L}+2\vec{S})$$
$$\vec{\mu}^2 = \left(\frac{e}{2mc}\right)^2[2\vec{J}^2 - \vec{l}^2 + 2\vec{S}^2]$$
$$< nljm_j|\vec{\mu}^2|nljm_j >= \left(\frac{e}{2m_jc}\right)^2[2j(j+1) - l(l+1) + \frac{3}{2}]\hbar^2$$
$$\Longrightarrow \mu = \left(\frac{e\hbar}{2m_jc}\right)\sqrt{2j(j+1) - l(l+1) + \frac{3}{2}}$$

From the projection theorem,we have

$$< nljm_j|\mu_0|nljm_j >= \frac{< nljm_j|\vec{J}\cdot\mu|nljm_j >}{j(j+1)\hbar^2} < jm_j|J_z|jm_j >= \frac{m_j}{j(j+1)\hbar^2} < jljm_j|\vec{J}\cdot\vec{\mu}|nljm_j >$$

$$\vec{J}\cdot\vec{\mu} = \frac{e}{2mc}[\vec{J}^2 + \vec{J}\cdot\vec{S}] = \frac{e}{2mc}[\frac{3}{2}\vec{J}^2 + \frac{1}{2}\vec{S}^2 - \frac{1}{2}\vec{L}^2]$$

Therefore,we have

$$< nljm_j|\vec{J}\cdot\mu|nljm_j >= \frac{e}{2mc}[\frac{3}{2}(j+1)j - \frac{1}{2}l(l+1) + \frac{3}{8}]\hbar^2$$
$$< nljm_j|\mu_z|nljm_j >= \frac{m_j}{j(j+1)}\frac{e}{2mc}[\frac{3}{2}(j+1)j - \frac{1}{2}l(l+1) + \frac{3}{8}]$$