Solution To PHYS622 Homework Set 7

1 Solution1

$S_z(S_z+\hbar)(S_z-\hbar)$ must be vanish for spin 1 particle since $m=0,1,-1$ are the eigenvalue of S_z ,and any spin state can be decomposed into the S_z eigenstates.Explicitly:

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \hbar$$

$$S_z + \hbar = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hbar$$

$$S_z - \hbar = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0\ \hbar \\ 0 & 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0\ \hbar \\ 0 & 0 & -2 \end{pmatrix} = 0$$

Similarly for

$$S_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}},$$

we can rotate it to the S_x eigenstate and get exactly same result as for S_z .

2 Solution2

$$D^{1/2}(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\beta/2 & e^{-i(\alpha-\gamma)/2}\sin\beta/2 \\ e^{i(\alpha-\gamma)/2}\sin\beta/2 & e^{i(\alpha+\gamma)/2}\cos\beta/2 \end{pmatrix}.$$

Compare with (3.2.45)

$$e^{-\frac{\vec{\delta}\cdot\hat{n}}{2}\theta} = \begin{pmatrix} \cos\theta/2 - i\hat{n}_z\sin\theta/2 & (-in_x - n_y)\sin\theta/2 \\ (-in_x + n_y)\sin\theta/2 & \cos\theta/2 + in_z\sin\theta/2 \end{pmatrix}$$

Therefore,we have

$$\cos\theta/2 = cos(\alpha+\gamma)/2\cos\beta/2$$
$$\sin\theta/2 = \sqrt{1-(\cos(\alpha+\gamma)/2\cos\beta/2)^2}$$
$$n_z\sin\theta/2 = \sin(\alpha+\gamma)/2\cos\beta/2$$
$$n_y\sin\theta/2 = \cos(\alpha-\gamma)/2\sin\beta/2$$
$$n_x sin\theta/2 = -sin(\alpha-\gamma)/2\sin\beta/2$$

3 Solution3

(a).

$J_+J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 + i(J_yJ_x - J_xJ_y) = \vec{J}^2 - J_z^2 + J_z\hbar$

(b).

$$J_-\psi_{j,m} = c_-\psi_{j,m-1}$$
$$J_+\psi^*_{j,m} = c^*_-\psi^*_{j,m-1}$$
$$< jm|J_+J_-|jm >= |c_-|^2 = [j(j+1) - m^2 + m]\hbar^2$$

4 Solution4

$$L_{x,y} = \frac{L_+ \pm L_-}{2}$$
$$< L_{x,y} >=< |L_+| > \pm < |L_-| >= 0$$

Since $L_\pm$ connect states with m to $m \pm 1$.

$$< lm|L_x^2|lm >= \frac{1}{4} < lm|L_+^2 + L_-^2 + L_+L_- + L_-L_+|lm >$$
$$= \frac{1}{4} < lm|L_+L_- + L_-L_+|lm >= \frac{1}{4} < lm|2L_+L_-2\hbar L_z|lm >$$
$$= \frac{\hbar^2}{2}[l(l+1) - m^2]$$

Similarly,we can get $< L_y^2 >=< L_x^2 >$.

5 Solution5

$$D(R) = e^{-iJ_y\beta/\hbar}$$

The probability amplitude of the new state in $|l, m >$ is

$$A_m =< l, m|D(R)|2, 0 >= D^2_{m,0}(\beta) = \sqrt{\frac{4\pi}{2l+1}}Y_2^{m*}(\beta, 0)$$
$$A_0 = \sqrt{\frac{4\pi}{5}}Y_2^0(\beta, 0) = \sqrt{\frac{4\pi}{5}}\sqrt{\frac{5}{16\pi}}(3\cos^3\beta - 1)$$
$$A_1 = \sqrt{\frac{4\pi}{5}}Y_2^1(\beta, 0) = -\sqrt{\frac{4\pi}{5}}\sqrt{\frac{15}{8\pi}}\sin\beta\cos\beta$$
$$A_2 = \sqrt{\frac{4\pi}{5}}Y_2^2(\beta, 0) = \sqrt{\frac{4\pi}{5}}\sqrt{\frac{15}{32\pi}}\sin^2\beta$$

So the probability:

$$
\begin{aligned}
|A_0|^2 &= \frac{1}{4}(3\cos^2\beta - 1)^2 \\
|A_1|^2 &= \frac{3}{2}(\sin\beta\cos\beta)^2 \\
|A_2|^2 &= \frac{3}{8}(\sin\beta)^4
\end{aligned}
$$

6 Solution6

(a).

$$J_y = \frac{1}{2i}(J_+ - J_-)$$

$< j, m'|J_y|1, m >$
$= \frac{\hbar}{2i}[\sqrt{j(j+1) - m(m+1)} < j, m'|j, m+1 > - \sqrt{j(j+1) - m(m-1)} < j, m'|j, m-1 >]$

Therefore

$$J_y = \frac{\hbar}{2i}\left[\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right] = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

(b).

$$e^{-iJ_y\beta/\hbar} = 1 - \frac{iJ_y\beta}{\hbar} + \frac{1}{2!}(\frac{iJ_y\beta}{\hbar})^2 - \frac{1}{3!}(\frac{iJ_y\beta}{\hbar})^3 + \cdots$$

$$
\begin{aligned}
J_y^2 &= -\frac{\hbar^2}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\
J_y^3 &= -\frac{-i\hbar^2}{2\sqrt{2}}\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} = \hbar^2 J_y
\end{aligned}
$$

So

$$J_y^4 = \hbar^2 J_y^2 \ldots$$

$$
\begin{aligned}
e^{-iJ_y\beta/\hbar} = 1 - \frac{i}{\hbar}J_y(\beta - \frac{\beta^3}{6} + \cdots) - \frac{J_y^2}{\hbar^2}(\frac{\beta^2}{2!} - \frac{\beta^4}{4!} + \cdots) \\
= 1 - \frac{i}{\hbar}J_y \sin\beta - (\frac{J_y}{\hbar})^2(1 - \cos\beta)
\end{aligned}
$$

(c).

$$d^{j=1}(\beta) = e^{-iJ_y\beta/\hbar} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{i}{\hbar} sin\beta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \frac{i\hbar}{\sqrt{2}} + \frac{1-\cos\beta}{2} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{-\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix}$$