Solution To PHYS622 Homework Set 6

1 Solution1

(a).

$$-\frac{\hbar^2}{2m}[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial^2}]=E\psi$$

Let

$$\psi(\rho,\theta,z)=[AJ_m(k\rho)+BN_m(k\rho)][Ce^{im\theta}+De^{-m\theta}][Ee^{\alpha z}+Fe^{-\alpha z}]$$

we have

$$m=integer, K\equiv\sqrt{\alpha^2-2m|E|/\hbar^2}$$

Because of the boundary conditions,we get

$$\alpha=ik, kL=l\pi, l=1,2,...$$

.

Also

$$\frac{d^2R}{dx}+\frac{1}{x}\frac{dR}{dx}+(1-m^2/x^2)R=0, where\ x=K\rho$$

Boundary conditions: $\psi(\rho=\rho_a)=\psi(\rho=\rho_b)=0$

$$AJ_m(k\rho_a)+BN_m(k\rho_a)=0$$
$$AJ_m(k\rho_b)+BN_m(k\rho_b)=0$$
$$\rightarrow J_m(K\rho_b)N_m(K\rho_a)-N_m(K\rho_b)J_m(K\rho_a)=0$$

Write the n-th root of equation as K_{mn} . Therefore

$$\alpha^2=-k^2=-l^2\pi^2/L^2=K^2+2m|E|/\hbar^2$$
$$\rightarrow E_{mn,l}=\frac{\hbar^2}{2m}(K_{mn}^2+l^2\pi^2/L^2)$$

(b).In case $\vec{A}=\frac{B\rho_a^2}{2\rho\hat{\theta}}$

$$H=\frac{1}{2m}(\vec{p}-\frac{e\vec{A}}{c})=-\frac{\hbar^2}{2m}(\nabla-\frac{ieB\rho_a^2}{2\hbar c}\hat{\theta})^2$$

Let $\nabla_\theta'=\frac{1}{\rho}(\frac{\partial}{\partial\phi}-\frac{ieB\rho_a^2}{2\hbar c})$ then $H=-\frac{\hbar^2}{2m}(\nabla')^2$. So

$$e^{im\phi}\rightarrow e^{im'\phi}, m'=m+\frac{eB\rho_a^2}{2\hbar c}$$
$$E_{m'n,l}=\frac{\hbar^2}{2m}(K_{m'n}^2+l^2\pi^2/L^2)$$

(c).For ground state

$$m^{'} = 0 \to m = -\frac{eB\rho_a^2}{2\hbar c} \equiv -N$$
$$\pi\rho_a^2 B = \frac{2\pi\hbar c N}{e}$$

2 Solution2

(a).

$$[\pi_i, \pi_j] = [p_i - \frac{eA_i}{c}, p_j - \frac{eA_j}{c}] = -[p_i, \frac{eA_j}{c}] + [p_j, \frac{eA_i}{c}] = \frac{i\hbar e}{c}([\partial_i, A_j] - [\partial_j, A_i]) = \frac{i\hbar e}{c}\varepsilon_{ijk}B_k$$

$$\frac{d\vec{x}}{dt} = (\vec{p} - \frac{e\vec{A}}{c})/m = \vec{\Pi}/m$$
$$\frac{d^2x_i}{dt^2} = \frac{d\Pi_i}{dt} = \frac{1}{i\hbar}[\Pi, H] = \frac{1}{i\hbar}([\Pi_i, \frac{\Pi^2}{2m}] + e[\Pi_i, \phi])$$

$$[\Pi_i, \phi] = -i\hbar\partial_i\phi = i\hbar E_i$$
$$\frac{1}{2m}[\Pi_i, \Pi^2] = \frac{1}{2m}[\Pi_i, \Pi_j]\Pi_j + \frac{1}{2m}\Pi_j[\Pi_i, \Pi_j]$$
$$= \frac{i\hbar e}{2cm}\varepsilon_{ijk}(B_k\Pi_j + \Pi_j B_k)$$
$$= \frac{i\hbar e}{2c}(\vec{P} \times \vec{B} - \vec{B} \times \vec{P})$$
$$= \frac{i\hbar e}{2c}(\frac{d\vec{x}}{dt} \times \vec{B} - \vec{B} \times \frac{d\vec{x}}{dt})$$

Therefore

$$m\frac{d^2\vec{x}}{dt} = \frac{d\vec{\Pi}}{dt} = e[E + \frac{1}{2c}(\frac{d\vec{x}}{dt} \times \vec{B} - \vec{B} \times \frac{d\vec{x}}{dt})]$$

(b).

$$[\frac{1}{2m}(-i\hbar\nabla - \frac{e\vec{A}}{c})^2 + e\phi]\psi = i\hbar\frac{\partial}{\partial t}\psi \tag{1}$$
$$[\frac{1}{2m}(i\hbar\nabla - \frac{e\vec{A}}{c})^2 + e\phi]\psi^* = -i\hbar\frac{\partial}{\partial t}\psi^* \tag{2}$$

$\psi^*(1) - \psi(2):$

$$i\hbar\frac{\partial}{\partial t}(\psi^*\psi) = \frac{1}{2m}[-\hbar^2(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) + \frac{2i\hbar e}{c}(\nabla \cdot \vec{A} + \vec{A} \cdot \nabla)(\psi^*\psi)]$$
$$\frac{\partial}{\partial t}(\psi^*\psi) = i\hbar\frac{1}{2m}\nabla \cdot (\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{e}{mc}\nabla \cdot (\vec{A}|\psi|^2)$$

Therefore,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\frac{\hbar}{m} Im(\psi^* \nabla \psi) - \frac{e}{mc}\vec{A}(\psi^* \psi)] = 0$$
$$\vec{J} = \frac{\hbar}{m} Im(\psi^* \nabla \psi) - \frac{e}{mc}\vec{A}(\psi^* \psi)$$

3 Solution3

(a).From 2(a). we get

$$[\Pi_x, \Pi_y] = \frac{i\hbar e}{c} B_z = i\hbar \frac{eB}{c}$$

(b).

$$H = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2 + P_z^2)$$
$$\frac{1}{2m}(\Pi_x^2 + \Pi_y^2 + P_z^2)\psi = E\psi$$

We can separate the motion in z-direction and get

$$\frac{1}{2m}(\Pi_x^2 + \Pi_y^2)\psi = (E - \frac{\hbar^2 k^2}{2m})\psi$$

For the motion in x-y plane,compare it with one-dimensional harmonic oscillation:

$$[\Pi_x, \Pi_y] = i\hbar \frac{eB}{c}, [p, x] = -i\hbar$$

The mapping is clear $\frac{k}{m} \rightarrow (\frac{eB}{mc})^2$ Therefore,

$$E_{n,k} = \frac{\hbar^2}{2m} + \frac{eB\hbar}{mc}(n + \frac{1}{2})$$

4 Solution4

$$\vec{\mu} = g_n(\frac{e\hbar}{2mc})\vec{S}, \quad L = g_n(\frac{e\hbar}{2mc})\vec{S} \cdot \vec{B}, \quad < \vec{S} \cdot \vec{B} >= \frac{1}{2}B$$

$$\Delta\phi = \frac{1}{\hbar}\int L dt = \frac{1}{\hbar} g_n \frac{e\hbar}{2mc}\frac{B}{2}T$$
$$T = \frac{l}{v} = \frac{l}{p/m} = \frac{m\lambda}{\hbar}$$

Therefore

$$\Delta\phi = g_n \frac{e\lambda l}{4\hbar c} B = \pi \rightarrow B = \frac{4\pi\hbar c}{e g_n \lambda l}$$

5 Solution5

$$U = \frac{a_0 + i\vec{\sigma}\cdot\vec{a}}{a_0 - i\vec{\sigma}\cdot\vec{a}} \equiv \frac{A}{A^+}$$

Note $[A, A^+] = 0$,i.e.they commute. So

$$U = A(A^+)^{-1}, U^+ = A^{-1}A^+$$

Therefore

$$UU^+ = A(A^+)^{-1}A^{-1}A^+ = A(AA^+)^{-1}A^+ = A(A^+A)^{-1}A^+ = AA^{-1}(AA^{-1})^+ = 1$$

$$detU = detA\ det(A^+)^{-1} = detA\ det(A^{-1}) = 1$$

We also can write U as explicitly form:

$$U = A(A^+)^{-1} = A^2(AA^+)^{-1} = \frac{a_0^2 + 2i(\vec{\sigma}\cdot\vec{a} - |\vec{a}|^2)}{a_0^2 + |\vec{a}|^2} = \frac{a_0^2 - |\vec{a}|^2)}{a_0^2 + |\vec{a}|^2} + \frac{2ia_0}{a_0^2 + |\vec{a}|^2}\vec{\sigma}\cdot\vec{a}$$

So

$$U = \cos\frac{\phi}{2} - i\vec{\sigma}\cdot\hat{a}\sin\frac{\phi}{2}$$
$$\hat{a} = -\hat{n}, \cos\frac{\phi}{2} = \frac{a_0^2 - |\vec{a}|^2)}{a_0^2 + |\vec{a}|^2}$$

6 Solution6

(a). $A \to 0$, $\frac{eB}{mc} \neq 0$

$$H \to \frac{eB}{mc}(S_z^{e^-} - S_z^{e^+})$$
$$H\chi_+^{e^-}\chi_-^{e^+} = \frac{eB}{mc}(\frac{\hbar}{2} + \frac{\hbar}{2})\chi_+^{e^-}\chi_-^{e^+} = \frac{eB\hbar}{mc}\chi_+^{e^-}\chi_-^{e^+}$$

Therefore $\chi_+^{e^-}\chi_-^{e^+}$ is an eigenfunction of H in the limit $A \to 0$ with eigenvalue $\frac{eB\hbar}{mc}$.

(b).

$$H = A\vec{S}^{e^-}\cdot\vec{S}e^+ = A\frac{(\vec{S}^{e^-} + \vec{S}^{e^+})^2 - (\vec{S}^{e^-})^2 - (\vec{S}^{e^+})^2}{2} = A\frac{(\vec{S}_{total})^2 - (\vec{S}^{e^-})^2 - (\vec{S}^{e^+})^2}{2}$$

$$(\vec{S}^{e^-})^2\chi_+^{e^-}\chi^{e^+} = \frac{3\hbar^2}{4}\chi_+^{e^-}\chi^{e^+}$$
$$(\vec{S}^{e^+})^2\chi_+^{e^-}\chi^{e^+} = \frac{3\hbar^2}{4}\chi_+^{e^-}\chi^{e^+}$$

But $\chi_+^{e^-}\chi^{e^+}$ is not the eigenstate of $\vec{S}_{total}$. However we can write

$$\chi_+^{e^-}\chi^{e^+} = \frac{1}{\sqrt{2}}(|1> +|0>)$$

where

$$\vec{S}^2_{total}|1>= s(s+1)\hbar|1>= 2\hbar|1>$$
$$\vec{S}^2_{total}|0>= s(s+1)\hbar|1>= 0$$

Therefore

$$< H >= -\frac{A\hbar^2}{4}$$