Solution To PHYS622 Homework Set 5

1 Solution1

(a).

$$\psi(x) = e^{if(x)/\hbar}$$
$$\hbar^2 \frac{\partial^2}{\partial x^2}\psi = -p^2\psi, \frac{p^2}{2m} = E - V(x),$$
$$i\hbar f'' - f'^2 + p^2 = 0.$$

(b).

$$f(x) = f_0(x) + \hbar f_1(x) + \hbar^2 f_2(x) + \dots$$
$$therefore f_0'^2 = p^2, if_0'' = 2f_0'f_1', if_1'' = 2f_0'f_1' + f_1'^2$$

(c).

$$f_0 = \int^x \sqrt{2m(E - V(x)}dx$$
$$f_1' = if_0''/2f_0'$$
$$Therefore \quad f_1 = i/2\ln(f_0') = i/4\ln(2m(E - V(x))),$$
$$\psi \sim \frac{1}{[E - V(x)]^2} e^{\pm\int^x \sqrt{2m(E-V(x))}dx}$$

2 Solution2

For a harmonic oscillator

$$V(x) = 1/2m\omega^2x^2$$

The WKB solution within the potential will should be

$$2C/\sqrt{p} = cos(-1/\hbar\int_x^{x_2} pdx + \pi/4)$$
$$2D/\sqrt{p} = cos(1/\hbar\int_x^{x_2} pdx - \pi/4)$$

These two solutions are consistent if C=D,and

$$-1/\hbar\int_x^{x_2} pdx - \pi/4 = 1/\hbar\int_x^{x_2} pdx - \pi/4 + n\pi$$
$$therefore \int_{x_1}^{x_2} \sqrt{2m(E - V(x))}dx = \hbar(n + 1/2)\pi, n = 0, 1, \dots$$

$$Let\ x = \sqrt{\frac{2E}{m\omega^2}}cos\theta, \int_0^\pi \frac{2E}{\omega}sin^2\theta d\theta = (n + \frac{1}{2})\hbar\pi \to \frac{2E}{\omega}\frac{\pi}{2} = (n + \frac{1}{2})\hbar\pi \to E_n = (n + \frac{1}{2}\hbar\omega).$$

3 Solution3

For a finite square barrier of height $V_0 > E$ and width $2a$,the WKB solution within the barrier is

$$\psi_{WKB} = \frac{A}{\sqrt{|p|}} e^{\frac{1}{\hbar}\int_0^x |p(x\prime)|dx\prime} + \frac{B}{\sqrt{|p|}} e^{-\frac{1}{\hbar}\int_0^x |p(x\prime)|dx\prime}$$

For approximation we can let $A = 0$,the solution near $x = 2a$ is

$$\psi(2a) \sim \frac{1}{\sqrt{|p|}} e^{-\frac{1}{\hbar}\int_0^{2a}\sqrt{2m(V_0-E)}}$$

$$T = \frac{|\psi(2a)|^2}{|\psi(0)|^2} \sim e^{-\frac{4a}{\hbar}\sqrt{2m(V_0-E)}}$$

4 Solution4

$$-\frac{\hbar^2}{2m}\psi\prime\prime - V_0\delta(x)\psi = -E\psi$$

There is one unique bound state

$$\psi(x) = Ae^{\pm\sqrt{\frac{2mE}{\hbar^2}}x}, x > 0, x < 0.$$

Because we have two boundary conditions:

$$\psi(x)|_{x=0+} = \psi(x)|_{x=0-}$$

$$\psi(x)\prime|_{x=0+} - \psi(x)\prime|_{x=0-} + \frac{2mV_0}{\hbar^2}\psi(0) = 0$$

$$Therefore\ E = V_0^2 m/2\hbar^2, A = \sqrt{mV_0/\hbar^2}.$$

If the potential is switched off at $t = 0$,the wavefunction for $t > 0$ will be

$$\psi(x, t > 0) = \int_{-\infty}^{+\infty} dx\prime\psi(x\prime, 0)K(x, x\prime; t, 0) = \sqrt{mV_0/\hbar^2}\sqrt{m/2\pi i\hbar t}\int_{-\infty}^{+\infty} dx\prime e^{-mV_0/\hbar^2 + i(x-x\prime)^2 m/2\hbar t}$$

5 Solution5

$$K(\vec{p}\prime\prime, \vec{p}\prime; t, t_0) = \sum_{a\prime} < \vec{p}\prime\prime|a\prime >< a\prime|\vec{p}\prime > e^{-\frac{iE_{(a\prime)}(t-t_0)}{\hbar}}$$

Take $a\prime$ to be p

$$K(\vec{p}\prime\prime, \vec{p}\prime; t, t_0) = \sum_p e^{-\frac{iE_{(p\prime)}(t-t_0)}{\hbar}}\delta(\vec{p}\prime\prime - \vec{p})\delta(\vec{p} - \vec{p}\prime) = e^{-\frac{p\prime^2(t-t_0)}{2m\hbar}}\delta(\vec{p}\prime\prime - \vec{p}\prime)$$

6 Solution6

Classical action

$$S = \int dt[\frac{m\dot{x}}{2} - V(x)]$$

$$S(n, n-1) = \Delta t[\frac{m}{2}(\frac{(x_n - x_{n-1})^2}{\Delta t} - \frac{\omega^2(x_n + x_{n-1})^2}{8})]$$

$$< x_n, t_n | x_{n-1}, t_{n-1} >= \sqrt{\frac{m}{2\pi\hbar\Delta t}} e^{\frac{iS(n,n-1)}{\hbar}}$$

The propagator is

$$K(x\prime\prime, t; x\prime, t_0) = \sqrt{\frac{m\omega}{2\pi i\hbar sin(\omega(t-t_0))}} e^{\frac{im\omega[(x\prime\prime^2 + x\prime^2) sin\omega(t-t_0) - 2x\prime x\prime\prime]}{2\hbar sin\omega(t-t_0)}}$$

In the limit $\Delta t \to 0$

$$sin\omega\Delta t \sim \omega\Delta t, cos\omega\Delta t \sim 1 - \omega^2\Delta t^2/2$$

$$therefore\ K \to \sqrt{\frac{m\omega}{2\pi i\hbar\Delta t}} e^{\frac{im}{2\hbar\Delta t}[(x_n - x_{n-1})^2 - \frac{2x_n^2\omega^2\Delta t^2}{2}]}$$

$$\sim \sqrt{\frac{m\omega}{2\pi i\hbar\Delta t}} e^{\frac{im}{2\hbar\Delta t}[(x_n - x_{n-1})^2 - \frac{\omega^2(x_n + x_{n-1})^2}{8}]}$$