Solution To PHYS622 Homework Set 3

JJS 1.21, 1.26, 1.29, 1.32, 1.33

1 Solution1

This is the one-dimensional rigid wall potential.The wave functions are $\psi_E(x) = \sqrt{2/a}sin(n\pi x/a), n = 1, 2, 3, ..., n = 1$ is the ground state $n > 1$ are the excited states. We have $<(\Delta x)^2>=<x^2>-<x>^2, <(\Delta p)^2>=<p^2>-<p>^2$,where $p = -i\hbar\partial_x$ and $p^2 = -\hbar^2\partial_x^2$ .For the rigid wall potential, we have

$$<x^2>=\frac{2}{a}\int_0^a x^2 sin^2(n\pi x/a)dx = a^2[1/3 - 1/2n^2\pi^2]$$

$$<x>=\frac{2}{a}\int_0^a x sin^2(n\pi/a)dx = a/2$$

$$<p^2>=\frac{2}{a}\int_0^a sin(n\pi x/a)(-\hbar^2\partial_x^2)sin(n\pi x/a)dx = \hbar^2/a^2(n\pi)^2$$

$$<p>=\frac{2}{a}\int_0^a sin(n\pi x/a)(-i\hbar\partial_x)sin(n\pi x/a)dx = 0.$$

Therefore the uncertainty product $<(\Delta x)^2><(\Delta p)^2>= \hbar^2/2[(n\pi)^2/6 - 1]$, for ground state $n = 1$,for excited states $n > 1$.

2 Solution2

We have $|S_x; +>= 1/\sqrt{2}(|+> +|->), |S_x; ->= 1/\sqrt{2}(|+> -|->)$. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ be the transformation matrix between S_z diagonal basis and S_x diagonal basis,i.e.

$$\begin{pmatrix} |S_x; +> \\ |S_x; -> \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} |+> \\ |-> \end{pmatrix} = U|r>$$

Obviously, $U_{11} = U_{12} = 1/\sqrt{2}$ and $U_{21} = 1/\sqrt{2}, U_{22} = -1/\sqrt{2}$.

Take $|S_x; +>= U_{11}|+> +U_{12}|->= \sum_a <a|S_x; +> |a>, |S_x; ->= U_{21}|+> +U_{12}|->= \sum_b <b|S_x; -> |b>$ with $a, b = +, -$ .Take the general form $U = \sum_r |b^{(r)}><a^{(r)}|$,then $U|r>= \sum_r |b^{(r)}><a^{(r)}|r>$. Identity $|a>, |b>$ with $|b^{(r)}>$ and $<a^{(r)}|r>$ with $<a|S_x; +>, <b|S_x; ->$ .We see that U can be expressed as $U = \sum_r |b^{(r)}><a^{(r)}|$.

3 Solution3

(a).We assume that $G(\vec{p})$ and $F(\vec{x})$ can be expressed as a power series

$$G(\vec{p}) = \sum_{n,m,l} a_{nml} p_i^n p_j^m p_k^l, F(\vec{x}) = \sum_{n,m,l} B_{nml} x_i^n x_j^m x_k^l.$$

By induction,we can prove $[x_i, p_i^n] = ni\hbar_i^{n-1}$ and $[p_i, x_i^n] = -ni\hbar_i^{n-1}$. So we could have

$$[x_i, p_i^n p_j^m p_k^l] = ni\hbar p_i^{n-1} p_j^m p_k^l, [p_i, x_i^n x_j^m x_k^l] = -ni\hbar x_i^{n-1} x_j^m x_k^l.$$

Using the series form for $G(\vec{p})$ and $F(\vec{x})$,we get

$$[x_i, G(\vec{p})] = i\hbar \partial G/\partial p_i, [p_i, F(\vec{x})] = -i\hbar \partial F/\partial x_i.$$

(b). $[x^2, p^2] = [x^2, p]p + p[x^2, p] = 2i\hbar xp + 2i\hbar px = 2i\hbar\{x, p\}$, The classical P.B. for $[x^2, p^2]_{cl} = \frac{\partial x^2}{x}\frac{\partial p^2}{p} - \frac{\partial x^2}{p}\frac{\partial p^2}{x} = 2x(2p) = 4xp$. Since in the classical limit $x, p = 2xp$, we have $[x^2, p^2]_{QM} = i\hbar[x^2, p^2]_{cl}$

4 Solution4

(a).Using $< x'|a >= \frac{1}{\sqrt{d\pi}} exp(ikx' - x'^2/2d^2)$, we have $\frac{\partial}{\partial x'} < x'|a >= \frac{1}{\sqrt{d\pi}}(ikx' - x'^2/2d^2)exp(ikx' - x'^2/2d^2)$ and $\frac{\partial^2}{\partial x'^2} < x'|a >= \frac{1}{\sqrt{d\pi}}(-k^2 - 1/d^2 - 2ikx'/d^2 + x'^2/d^4)exp(ikx' - x'^2/2d^2)$. So we have

$$< p >= \int_{-\infty}^{+\infty} < a|x' > (-i\hbar\frac{\partial}{\partial x'}) < x'|a > dx' = \hbar k.$$

$$< p^2 >= \int_{-\infty}^{+\infty} < a|x' > (-\hbar^2\frac{\partial^2}{\partial x'^2}) < x'|a > dx' = \hbar^2/2d^2 + \hbar^2 k^2.$$

(b). $< p|a >= \frac{\sqrt{d}}{\sqrt{\hbar\pi}} exp[-(p - \hbar k)^2 d^2/2\hbar^2]$. Therefore

$$< p >= \int_{-\infty}^{+\infty} < a|p > p < p|a > dp = \hbar k,$$

$$< p^2 >= \int_{-\infty}^{+\infty} < a|p > p^2 < p|a > dp = \hbar^2/2d^2 + \hbar^2 k^2.$$

5 Solution5

(a).(i)First note

$$< p'|x|p'' >= \int < p'|x|x' >< x'|p'' > dx'$$

$$= \int x' < p'|x' >< x'|p'' > dx' = \frac{1}{2\pi\hbar}\int dx' x' e^{-ix'(p'-p'')/\hbar}$$

But $\delta(p'-p'') = \frac{1}{2\pi\hbar}\int dx' x' e^{(-ix'(p'-p'')/\hbar}$, so $\frac{\partial}{p'}\delta(p'-p'') = \frac{1}{2\pi\hbar}\int dx' \frac{x'}{i\hbar} e^{-ix'(p'-p'')/\hbar}$. Therefore, $< p'|x|p'' >= i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')$. On the other hand,$< p'|x|a >= \int dp'' < p'|x|p'' >< p''|a >= \int dp'' i\hbar\frac{\partial}{\partial p'} < p''|a >= i\hbar\frac{\partial}{\partial p'} < p'|a >$. For(ii)we can perform an analogous procedure. $< \beta|x|\alpha >= \int dp' < \beta|p' >< p'|x|a >= \int < \beta|p' > i\hbar\frac{\partial}{\partial p'} < p'|a >= \int dp' \phi^*_\beta(p') i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$.

(b).Consider momentum eigenket with eigenvalue p' , $p|p' >= p'|p' >$. Now let's consider the ket $|p', \Xi >= exp[ix\Xi/\hbar]|p' >$. We have

$$p|p', \Xi >= pexp[ix\Xi/\hbar]|p' >= exp[ix\Xi/\hbar]p|p' > +[p, exp[ix\Xi/\hbar]]|p' >$$

And we know $[p, exp[ix\Xi/\hbar]] = -i\hbar(i\Xi/\hbar exp[ix\Xi/\hbar])$. So

$$\begin{aligned} p|p', \Xi >= pexp[ix\Xi/\hbar]|p' >&= exp[ix\Xi/\hbar]p'|p' > +\Xi exp[ix\Xi/\hbar]|p' > \\ &= (p' + \Xi)exp[ix\Xi/\hbar] = (p' + \Xi)|p', \Xi > . \end{aligned}$$

Therefore $|p', \Xi >$ is eigenket of p with eigenvalue $p' + \Xi$ and operator $exp[ix\Xi/\hbar]$ is omentum translation operator and x is the generator of momentum translations.