Solution To PHYS622 Homework Set 2

1 Solution1

(a).The characteristic equation $det[B-\lambda I]=0$,leads to $(\lambda-b)^2(\lambda+b)=0$.Hence $\lambda=\pm b$ and $\lambda=b$ is a two-fold degenerate eigenvalue.
(b).Straightfowrd matrix multiplication gives

$$AB=\begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}=BA\rightarrow[A,B]=0.$$

(c).The eigenvectors of B,together with $[A,B]=0$,yield simultaneous eigenvectors of $A$and B. Let $\lambda_i$be eigenvalues of B, and corresponding eigenvectors are

$$u^i=\begin{pmatrix} u_1^i \\ u_2^i \\ u_3^i \end{pmatrix},where\ Bu^i=\lambda_i u^i,i=1,2,3$$

For $\lambda_1=b$,we could choose eigenvector as

$$u^1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}=|1>.$$

For$\lambda_2=b$,the eigenvector could be

$$u^2=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}=\frac{1}{\sqrt{2}}(|1>+i|3>),where\ |2>=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},|3>=\begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}.$$

Note u^2can not be chosen as same as u^1 although they have same eigenvalue,because they should be orthogonal.

For nondegenerate $\lambda_3=-b$, againu^3must be orthogonal to u^1,u^2. We can have

$$u^2=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}=\frac{1}{\sqrt{2}}(|1>-i|3>).$$

In this new set $u^i(i=1,2,3)$,obviously $Au^1=au^1$, $Au^2=-au^2$, $Au^3=-au^3$.

2 Solution2

Choosing the S_z diagonal basis, the first measurement corresponds to the operator $M(+)=|+><+|$.The secod measurement is expressed by the operator$M(+;\hat{n})=$

$|+;\hat{n}><+;\hat{n}|$where $|+;\hat{n}>= cos(\beta/2)|+> +sin(\beta/2)|->$ with $\alpha = 0$. Therefore

$$M(+;\hat{n}) = (cos(\beta/2)|+> +sin(\beta/2)|->)(cos(\beta/2)<+| + sin(\beta/2)<-|)$$
$$= cos^2(\beta/2)|+><+| + cos(\beta/2)sin(\beta/2)(|+><-| + |-><+|) + sin^2(\beta/2)|-><-|.$$

The final measurement corresponds to the operator$M(-) = |-><-|$,and the total measurement

$$M_T = M(-)M(+;\hat{n})M(+) = cos(\beta/2)sin(\beta/2)|-><+|$$

the intensity of the final $S_z = \frac{-\hbar}{2}$beam, when the $S_z = \frac{\hbar}{2}$beam surviving the first measuremenbt is normalized to unity, is thus $cos^2(\beta/2)sin^2(\beta/2) = sin^2(\beta)/4$.To maximize $S_z = \frac{-\hbar}{2}$ final beam, set $\beta = \pi/2$,i.e. along OX, and intesity is 1/4.

3 Solution3

(a).$H$is a diagonal matrix in the orhtonomal basis spanned by $|u_1>, |u_2>, |u_3>$,with the eigenvalues $\hbar\omega, 2\hbar\omega, 2\hbar\ \omega$ respectively. When we measure the energy of system,we can get $\hbar$ with the probability $(\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$,and $2\hbar$ with the probability $(\frac{1}{2})^2+(\frac{1}{2})^2 = \frac{1}{2}$ .Since $<H>=<\psi|H|\psi>= \frac{3}{2}\hbar\omega$,and $<H^2>=<\psi|H^2|\psi>= \frac{5}{2}(\hbar\omega)^2$, we can get $\Delta H = \sqrt{<H^2> - <H>^2} = \frac{1}{2}\hbar\omega$.
(b).Solve characteristic equation $det[A - \lambda I] = 0$,we get $\lambda_1 = a, \lambda_2 = a$ and $\lambda_3 = -a$. We can choose the eigenstates as $|\psi_1>= |u_1>, |\psi_2>= \frac{1}{\sqrt{2}}(|u_2> +|u_3>), |\psi_2>= \frac{1}{\sqrt{2}}(|u_2> -|u_3>)$ respectively. So the possible measurement value is a with probability $|<\psi_1|\psi>|^2 + |<\psi_2|\psi>|^2 = 1$. The probability to get value $-a$ is 0,because $|<\psi_3|\psi>|^2 = 0$. After measurement,the state vector could be one of $|\psi_1>, |\psi_2>, \frac{1}{\sqrt{2}}(|\psi_1> +|\psi_2>)$.
(c).Since $<B>=<\psi|B|\psi>= (\frac{1}{\sqrt{2}} + \frac{1}{4})b$,and $<B^2>=<\psi|B^2|\psi>= b^2$, we can get $\Delta B = \sqrt{<B^2> - <B>^2} = b\frac{\sqrt{7-4\sqrt{2}}}{4}$.

4 Solution4

Take the normalized linear combination$|>= \alpha|+> +(1-\alpha^2)^{\frac{1}{2}}e^{i\beta}|->$,where α is real and $|\alpha| \leq 1$. Then elementary calculations yield $<|(\Delta S_x)^2|>= \frac{\hbar^2}{4}[1 - 4\alpha^2(1-\alpha^2)cos^2(\beta)]$ and $<|(\Delta S_y)^2|>= \frac{\hbar^2}{4}[1 - 4\alpha^2(1-\alpha^2)sin^2(\beta)]$. And the produce is

$$<|(\Delta S_x)^2|><|(\Delta S_y)^2|>= \frac{\hbar^4}{16}[1 - 4\alpha^2(1-\alpha^2) + 4\alpha^4(1-\alpha^2)^2 sin^2(2\beta)].$$

Maximum for $sin^2(2\beta)$is when $\beta = \pi/4$,then$<|(\Delta S_x)^2|><|(\Delta S_y)^2|>= \frac{\hbar^4}{16}[1 - 2\alpha^2(1-\alpha^2)]^2$ .It is clear that $\alpha^2 = 1/2$ is minimum,and when $\alpha^2 =$

$0, 1$ the maximum value $\hbar^4/16$ is reached. Hence the linear 'combination' that maximizes uncertainty product is $e^{i\pi/4}|- >$ or$\pm|+ >$. That $\pm|+ >$ does not violate uncertainty relation is obvious because of $[S_x, S_y] = i\hbar S_z$ and $< |S_z| >= \hbar/2$. For the $e^{i\pi/4}|- >$ case, we note that the phase factor $e^{i\pi/4}$ cancels out in the scalar product, and $< -|S_x|- >=< -|S-y|- >= 0$ while $< -|S_x^2|- >=< -|S_y^2|- >= \hbar^2/4$. Again $< -|[S_x, S_y]|- >=< -|i\hbar S-z|- >= i\hbar(-\hbar/2) = -i\hbar^2/2$ .Therefore we have $< |(\Delta S_x)^2| >< |(\Delta S_y)^2| >= \hbar^4/16 = 1/4| < -|[S_x, S_y]|- > |^2$, no violation.

5 Solution5

(a).We calculate this problem with brute force $\rho_A = Tr_B(|\psi >< \psi|) =_B< +|\psi >< \psi|+ >_B +_B < -|\psi >< \psi|- >_B$. Substitute $|\psi >= \frac{1}{\sqrt{2}}|+ >_A (\frac{1}{2}|+ > +\frac{\sqrt{3}}{2}|- >_B) + \frac{1}{\sqrt{2}}|- >_A (\frac{\sqrt{3}}{2}|+ > +\frac{1}{2}|- >_B)$ into above equation,we can get $\rho_A = \frac{1}{2}{}_A|+ >< +|_A + \frac{1}{2}{}_A|- >< -|_A + \frac{\sqrt{3}}{4}{}_A|- >< +|_A + \frac{\sqrt{3}}{4}{}_A|+ >< -|_A$ or in matrix form $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{pmatrix}$.With same procedure,$\rho_B = \frac{1}{2}{}_B|+ >< +|_B + \frac{1}{2}{}_B|- >< -|_B + \frac{\sqrt{3}}{4}{}_B|- >< +|_B + \frac{\sqrt{3}}{4}{}_B|+ >< -|_B$ or $\rho_B = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{pmatrix}$in matrix form.(In this problem,one should be careful. Do not mix the eigenkets $|\pm >_A, |\pm >_B$.)

(b).We need to diagnolize ρ_A.From the characteristic equation $det \begin{pmatrix} \frac{1}{2} - \lambda & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} - \lambda \end{pmatrix} = 0$, we have eigenvalues $\lambda_\pm = \frac{1}{2} \pm \frac{\sqrt{3}}{4}$ .The eigenstates can written as $|\psi_A^\pm >= \frac{1}{\sqrt{2}}(|+ >_A +|+ >_A)$. Hence $|\phi_B^\pm >=< \psi_A^\pm|\psi >= 1\frac{1\pm\sqrt{3}}{4}$,and normalize it,one can have $< \phi_B^\pm|\phi_B^\pm >= 1 \pm \frac{\sqrt{3}}{2} \equiv P_\pm$.Write normalized $|\psi >$ as $|\psi_B^\pm >= \frac{1}{\sqrt{P_\pm}}|\phi_B^\pm >$,so the Schmidt decomposition of $|\psi >$ is $|\psi >_{AB}= \sqrt{P_+}|\psi^+ >_A |\psi^+ >_B +\sqrt{P_-}|\psi^- >_A |\psi^- >_B$.

6 Solution6

Method 1. Use the brute force as in Problem5. Note $|+ >_m= cos(\frac{\theta_m}{2})e^{-i\frac{\phi_m}{2}}|+ > +sin(\frac{\theta_m}{2})e^{-i\frac{-\phi_m}{2}}|- >$ and $|+ >_n= cos(\frac{\theta_n}{2})e^{-i\frac{\phi_n}{2}}|+ > +sin(\frac{\theta_n}{2})e^{-i\frac{-\phi_n}{2}}|- >$ .The probability that both spins are up along their respective axis is $P =< +|_m < +|_n\rho|+ >_n |+ >_m= \frac{1}{8} + \frac{1}{8}[1 - (cos\theta_m cos\theta_n + sin\theta_m sin\theta_n cos(\theta_m - \theta_n)]$.To go further, we need to use the condition $\vec{m} \cdot \vec{n} = cos\theta$ .Write the unit vectors $\vec{m}$ and $\vec{n}$ as components form $\vec{m} \cdot \vec{n} = (sin\theta_m cos\phi_m \vec{e}_x + sin\theta_m sin\phi_m \vec{e}_y + cos\theta_m \vec{e}_z)(sin\theta_n cos\phi_n \vec{e}_x + sin\theta_n sin\phi_n \vec{e}_y + cos\theta_n \vec{e}_z) = cos\theta_m cos\theta_n + sin\theta_m sin\theta_n cos(\theta_m - \theta_n) = cos\theta$ .Therefore,$P = \frac{1}{4} - \frac{1}{8}cos\theta$.

Method 2. If we write ρ as matrix form,the calculation can be simplified.

$$\rho = \begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 \\ 0 & \frac{3}{8} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

We can write the measurement operator as $A = |+>_n<+|_n(|+>_m<+|_m)$,where $|+>_m = cos\frac{\theta}{2}|+> + sin\frac{\theta}{2}|->$,and $|+>_n = |+>$. In matrix form

$$\rho = \begin{pmatrix} cos^2\frac{\theta}{2} & cos\frac{\theta}{2}sin\frac{\theta}{2} & 0 & 0 \\ cos\frac{\theta}{2}sin\frac{\theta}{2} & sin^2\frac{\theta}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$$P = Tr(\rho A) = Tr \begin{pmatrix} \frac{cos^2\frac{\theta}{2}}{8} & \frac{sin\frac{\theta}{2}cos\frac{\theta}{2}}{8} & 0 & 0 \\ \frac{3sin\frac{\theta}{2}cos\frac{\theta}{2}}{8} & \frac{3sin^2\frac{\theta}{2}}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{cos^2\frac{\theta}{2}}{8} + \frac{3sin^2\frac{\theta}{2}}{8} = \frac{1}{4} - \frac{1}{8}cos\theta.$$