Solution To PHYS622 Homework Set 10

1 Solution1

(a).

$$T_{\vec{d}}\psi(\vec{x}) = \psi(\vec{x}+\vec{d})$$
$$T_{\vec{d}'}T_{\vec{d}}\psi(\vec{x}) = \psi(\vec{x}+\vec{d}'+\vec{d})$$
$$T_{\vec{d}}T_{\vec{d}'}\psi(\vec{x}) = \psi(\vec{x}+\vec{d}+\vec{d}')$$
$$\Longrightarrow [T_{\vec{d}}, T_{\vec{d}'}] = 0$$

(b). $D(\hat{n},\phi)$ does not commute with $D(\hat{n}',\phi')$.
(c). $T_{\vec{d}}$ and Π do not commute.

$$\Pi T_{\vec{d}}\psi(\vec{x}) = \psi(-\vec{x}-\vec{d})$$
$$T_{\vec{d}}\Pi\psi(\vec{x}) = \psi(-\vec{x}+\vec{d})$$

(d).

$$\Pi D(\hat{n},\phi)\psi(\vec{x}) = \Pi\psi(\vec{x}') = \psi(-\vec{x}'), \vec{x}' = D(\hat{n},\phi)\vec{x}.$$

On the other hand

$$D(\hat{n},\phi)\Pi\psi(\vec{x}) = D(\hat{n},\phi)\psi(-\vec{x}) = \psi(-\vec{x}')$$

They commute.

2 Solution2

(a).

$$y_l^{j=l\pm 1/2,m} = \frac{1}{\sqrt{2l+1}}\begin{pmatrix} \pm\sqrt{l\pm m+1/2}Y^{m-1/2} \\ \sqrt{l\mp m+1/2}Y^{m+1/2} \end{pmatrix}$$
$$y_{l=0}^{j=l/2,m=1/2} = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(b).

$$\vec{\sigma}\cdot\vec{x}\frac{1}{\sqrt{4\pi}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{r}{\sqrt{2\pi}}\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix}$$
$$= -r\begin{pmatrix} -Y_1^0(\theta,\phi)/\sqrt{3} \\ Y_1^1(\theta,\phi)/\sqrt{2/3} \end{pmatrix} = -ry_{l=1}^{j=1/2,m=1/2}$$

(c).$\vec{S}\cdot\vec{x}$ is a scalar(spherical tensor of rank 0)under rotation,hence by Wigner-Eckart theorem it cannot change j and m.But under space inversion $\vec{S}\cdot\vec{x}$ is odd.So $\vec{S}\cdot\vec{x}$ connects even parity with odd parity,and we note $l=0$ and $l=1$ have opposite parity.

3 Solution3

Symmetric states

$$\begin{aligned}\psi_I &= A\sin(k_s(x+a+b))\\ \psi_{II} &= B\cosh(\kappa_s x)\\ \psi_{III} &= -A\cosh(\kappa_s(x-a-b))\end{aligned}$$

Antisymmetric states

$$\begin{aligned}\phi_I &= C\sin(k_s(x+a+b))\\ \phi_{II} &= D\cosh(\kappa_s x)\\ \phi_{III} &= C\cosh(\kappa_s(x-a-b))\end{aligned}$$

All of states should satisfy Schrodinger's equations and appropriate boundary conditions.Matching the solutions and derivatives at each boundary we could have

$$\begin{aligned}k_s &= \frac{\pi}{b+\coth\kappa a/\kappa}\\ k_a &= \frac{\pi}{b+\tanh\kappa a/\kappa}\end{aligned}$$

The lowest lying states are

$$\begin{aligned}E_s &= \frac{\hbar^2 k_s^2}{2m} = \frac{\hbar^2\pi^2}{2m(b+\coth\kappa a/\kappa)^2}\\ E_a &= \frac{\hbar^2 k_a^2}{2m} = \frac{\hbar^2\pi^2}{2m(b+\tanh\kappa a/\kappa)^2}\end{aligned}$$

Therefore,

$$\Delta = E_a - E_s = \frac{\hbar^2\pi^2}{2mb^2[(1+\tanh\kappa a/\kappa b)^2-(1+\coth\kappa a/\kappa b)^2]}$$

4 Solution4

(a).

$$\psi(\vec{x},t) = e^{i(\vec{p}\cdot\vec{x}/\hbar-\omega t)}$$

$$\psi^*(\vec{x},-t) = e^{-i(\vec{p}\cdot\vec{x}/\hbar+\omega t)} = e^{i(-\vec{p}\cdot\vec{x}/\hbar-\omega t)}$$

(b).

$$\vec{\sigma}\cdot\hat{n}\chi_+(\hat{n}) = \chi_+(\hat{n}), \chi_+ = \begin{pmatrix} \cos(\beta/2)e^{-i\alpha/2} \\ \sin(\beta/2)e^{i\alpha/2} \end{pmatrix}$$

$$-i\sigma_2\chi_+^*(\hat{n}) = -i\sigma_2 \begin{pmatrix} \cos(\beta/2)e^{i\alpha/2} \\ \sin(\beta/2)e^{-i\alpha/2} \end{pmatrix} = \begin{pmatrix} -\sin(\beta/2)e^{i\alpha/2} \\ \cos(\beta/2)e^{-i\alpha/2} \end{pmatrix}$$

Consider

$$\chi_+(\hat{n})|_{\beta\to\pi-\beta,\alpha\to\pi+\alpha} = \begin{pmatrix} -\sin(\beta/2)e^{i\alpha/2} \\ \cos(\beta/2)e^{-i\alpha/2} \end{pmatrix} = -i\sigma_2\chi_+^*(\hat{n})$$

Therefore

$$\chi_-(\hat{n}) = \chi_+(\hat{n})|_{\beta\to\pi-\beta,\alpha\to\pi+\alpha} = -i\sigma_2\chi_+^*(\hat{n})$$

5 Solution5

(a).

$$\Theta D(R)|j,m>=\Theta e^{-i\vec{J}\cdot\hat{n}\theta/\hbar}|j,m>=\Theta e^{-i\vec{J}\cdot\hat{n}\theta/\hbar}\Theta^{-1}\Theta|j,m>$$

$$\Theta e^{-i\vec{J}\cdot\hat{n}\theta/\hbar}\Theta^{-1} = e^{+i(-\vec{J})\cdot\hat{n}\theta/\hbar} = e^{-i\vec{J}\cdot\hat{n}\theta/\hbar}, \quad \Theta|j,m>=(-1)^m|j,-m>$$

Therefore

$$\Theta D(R)|j,m>=\Theta e^{-i\vec{J}\cdot\hat{n}\theta/\hbar}|j,m>=(-1)^m D(R)|j,-m>$$

(b).

$$\Theta D(R)|j,m>=\Theta[\sum_{m'} D(R)_{m',m}|j,m>]$$

$$L.H.S = (-1)^m D(R)|j,-m>=(-1)^m\sum_{m'} D(R)_{-m',-m}|j,-m'>$$

$$R.H.S = \sum_{m'} D^*_{m',m}\Theta|j,m'>=\sum_{m'} D^*_{m',m}(R)(-1)^{m'}|j,-m'>$$

So we have

$$D^*_{m',m}(R) = (-1)^{m-m'}D_{-m',-m}(R)$$

(c). For $j = 1/2$

$$\Theta|j,m>=\eta e^{-i\pi J_y/\hbar}K|jm>=\eta e^{-i\pi J_y/\hbar}|jm>=\eta\sum_{m'} D_{m',m}(\pi)|j,m'>$$

$$=\eta\sum_{m'}\delta_{m',-m}(-1)^{j+m}|j,m'>=\eta(-1)^{j+m}|j,-m>$$

Choose $\eta = (-1)^{-j}$

$$\Theta|j,m>=(-1)^m|j,-m>=(i)^{2m}|j,-m>$$

6 Solution6

$$H = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

Eigenvalues and eigenstates

$$\begin{aligned} E = 0, & \qquad |1,0> \\ E = \hbar^2(A+B), & \frac{1}{\sqrt{2}}(|1,+1> + |1,-1>) \\ E = \hbar^2(A-B), & \frac{1}{\sqrt{2}}(|1,+1> - |1,-1>) \end{aligned}$$

Assume A, B are real

$$\Theta H \Theta^{-1} = H$$

Because of

$$\Theta S_{x,y,z}^2 \Theta^{-1} = (-S_{x,y,z})^2 = S_{x,y,z}^2.$$

Therefore,H is invariant under time reversal.

$$\begin{aligned} \Theta|1,0> &= |1,0> \\ \Theta\frac{1}{\sqrt{2}}(|1,+1> + |1,-1>) = \frac{1}{\sqrt{2}}(-|1,-1> - |1,+1>) &= -\frac{1}{\sqrt{2}}(|1,+1> + |1,-1>) \\ \Theta\frac{1}{\sqrt{2}}(|1,+1> - |1,-1>) = \frac{1}{\sqrt{2}}(-|1,-1> + |1,+1>) &= \frac{1}{\sqrt{2}}(|1,+1> - |1,-1>) \end{aligned}$$

All states are invariant under time reversal.