Solution To PHYS622 Homework Set 1

1 Solution1

$|L>=\frac{1}{\sqrt{2}}(|e_x>-i|e_y>)$,$|R>=\frac{1}{\sqrt{2}}(|e_x>+i|e_y>)$

$P_L=|L><L|=\frac{1}{2}(|e_x>-i|e_y>)(|e_x>+i|e_y>)$

$=\frac{1}{2}(|e_x><e_x|-i|e_y><e_x|+i|e_x><e_y|+|e_y><e_y|)$

If we set $|e_x>=\binom{1}{0}$, $|e_y>=\binom{0}{1}$,$<e_x|=(1,0)$,$<e_y|=(0,1)$

$P_L=\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$

Similarly

$P_R=\frac{1}{\sqrt{2}}\binom{1}{i}(1,-i)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$

In the same basis,

$$P_x=|e_x><e_x|=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$P_y=|e_y><e_y|=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

Intensity after passing through X,R then Y polarizer should be

$|<e_y|P_Y P_R P_X|e_x>|^2=\frac{1}{4}$

2 Solution2

(a). $tr(XY)=\sum_i <i|XY|i>=\sum_{i,j}<i|X|j><j|Y|i>=\sum_{j,i}<j|X|i><i|Y|j>=\sum_j <j|YX|i>=tr(XY)$.

(b). $<i|XY|j>=<X^\dagger i|Y|j>=<Y^\dagger X^\dagger i|j>$,
and also $<i|XY|j>=<(XY)^\dagger i|j>$, thus $(XY)^\dagger=Y^\dagger X^\dagger$.

(c). $A|i>=a_i|i>$ for all i, thus $e^{if(A)}=\sum_{n=0}\frac{[if(A)]^n}{n!}|i>=\sum_{n=0}\frac{[if(a_i)]^n}{n!}|i>=e^{if(a_i)}|i>$.

(d). $\sum_{a'}\psi_{a'}(\vec{x}')\psi_{a'}(\vec{x}'')=\sum_{a'}<a'|\vec{x}'><\vec{x}''|a'>=\sum_{a'}<\vec{x}'|a'><a'|\vec{x}''>=<\vec{x}''|\vec{x}'>$.

3 Solution3

(a). $|\alpha><\beta|=\sum_{a',a''}|a'><a'|\alpha><\beta|a''><a''|=\sum_{a',a''}|a'><a''|\times(<a'|a><a''|\beta>^*)$.Hence $|\alpha><\beta|=[<a^{(i)}|\alpha><a^{(j)}|\beta>^*]$,where expression inside square bracket is the (i,j)matrix element.

(b). $|\alpha>=|S_z=\hbar/2>=|+>$, $|\beta>=|S_x=\hbar/2>=\frac{1}{2}(|+>+|->)$,

so $|\alpha><\beta| = \frac{1}{\sqrt{2}}(|+><+|+|+><-|)$
if $|+>= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|->= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, *then* $|\alpha><\beta| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$.

4 Solution4

The orthonormality property is$<+|+>=<-|->=1, <+|->=<-|+>=0$.Hence using the explicit representations of $S_i$in terms of the linear combinations of bra-ket products:$S_x = \frac{\hbar}{2}(|+><-|+|-><+|)$, $S_y = \frac{\hbar}{2}(-i|+><-|+i|-><+|)$, $S_z = \frac{\hbar}{2}(|+><+|-|-><-|)$, we obtain by elementary calculation $[S_i, S_j] = i\epsilon_{ijk}\hbar S_k$and $S_i, S_j = \hbar^2/2\delta_{ij}$.

5 Solution5

Rewrite Has $H = \frac{1}{2}(H_{11}+H_{22})(|1><1|+|2><2|) + \frac{1}{2}(H_{11}-H_{22})(|1><1|-|2><2|)+H_{12}(|1><2|+|2><1|)$,where the three operator terms on r.h.s behave like I,S_z,S_x respectively. Because the identity operator I remains the same under any change of basis, we ignore the$\frac{1}{2}(H_{11}+H_{22})(|1><1|+|2><2|)$ for moment. Compare now with the spin $\frac{1}{2}$ problem.$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}n_x(|+><-|+|-><+|) + \frac{\hbar}{2}n_y(-i|+><-|+i|-><+|) + \frac{\hbar}{2}n_z(|+><+|-|-><-|)$. The analogy is :$\frac{1}{2}n_x \to H_{12}, \frac{1}{2}n_y \to 0, \frac{1}{2}n_z \to \frac{1}{2}(H_{11}-H_{22})$. So one of the energy eigenkets is $cos(\beta/2)|1> + sin(\beta/2)|2>$where β,analogous to $tan^{-1}(n_x/n_z)$,is given by $\beta = tan^{-1}[\frac{2H_{12}}{H_{11}-H_{22}}]$. The other energy eigenket can be written down by the orthogonality requirement(or by letting$\beta \to \beta\pi$)as $-sin(\beta/2)|1> + cos(\beta/2)|2>$. The energy eigenvalues can be obtained by diagonalizing

$$\begin{pmatrix} \frac{1}{2}(H_{11}-H_{22}) & H_{12} \\ H_{12} & \frac{-1}{2}(H_{11}-H_{22}) \end{pmatrix}.$$

But they can also be obtained by comparing with the sipn $\frac{1}{2}$ problem:

$$(\frac{\hbar}{2}n_x)^2 + (\frac{\hbar}{2}n_z)^2 = \frac{\hbar^2}{4} \to eigenvalue\frac{\hbar}{2}$$

so by analogy the eigenvalue in our case is $[\frac{1}{4}(H_{11}-H_{22})^2 + H_{12}^2]^{1/2}$.We must still add to this the $\frac{1}{2}(H_{11}+H_{22})$. The final answer is$\frac{1}{2}(H_{11}+H_{22}) \pm [\frac{1}{4}(H_{11}-H_{22})^2 + H_{12}^2]^{1/2}$. where $\pm$ is the analogue of parallel(anti-parallel)spin direction to $\hat{n}$.For $H_{12}=0$,we get $\beta = 0$ *or* π.The eigenvalues are just H_{11}, H_{22}.

6 Solution6

Here $\vec{S}\cdot\hat{n}|\hat{n};+>=\frac{\hbar}{2}|\hat{n};+>$ and $|\hat{n};+>=cos(\gamma/2)|+>+sin(\gamma/2)|->=\begin{pmatrix} cos(\gamma/2) \\ sin(\gamma/2) \end{pmatrix}$.It is easily seen that the eigenket of $S_x$belonging to eigenvalue $+\hbar/2$,is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.Thus (a)the probability of getting $+\hbar/2$when $S_x$is measured is $|\frac{1}{\sqrt{2}}(1,1)\begin{pmatrix} \cos(\gamma/2) \\ sin(\gamma/2) \end{pmatrix}|^2$.(b) $<S_x>=\frac{\hbar}{2}(cos(\gamma/2),sin(\gamma/2))\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} cos(\gamma/2) \\ sin(\gamma/2) \end{pmatrix}=\frac{\hbar}{2}sin\gamma$.Hence $<(S_x-<S-x>)^2>=<S_x^2>-<S_x>^2=\hbar^2/4-(\hbar^2/4)cos^2\gamma$. Answers are entirely reasonable for $\gamma=0,\pi$(paralled and anti-parallel to OZ),and for $\gamma=\pi/2$(along OX).