Department of Physics
University of Maryland, College Park

Midterm Exam, Physics 622 — Friday, Nov. 11, 2005

Problem 1: Particle in Magnetic Field (40 pts)

1) Consider the expectation value of a free-particle hamiltonian

$$\langle H\rangle = \int d^3\vec{r}\psi^*(\vec{r})\left(-\frac{\hbar^2\nabla^2}{2m}\right)\psi(\vec{r}) \tag{1}$$

Show that it is invariant under a constant phase transformation

$$\psi(x) \to e^{i\theta}\psi(x) \tag{2}$$

where θ is a constant. Also show that it is not invariant if θ depends on coordinate $\vec{r}$. (10pts)

2) Following 1), if one introduces the vector potential $\vec{A}$ by modifying the hamiltonian,

$$\langle H\rangle = \int d^3\vec{r}\psi^*\left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e\vec{A}}{c}\right)^2\right)\psi \tag{3}$$

the expectation value can be made invariant under the transformation $\psi(x) \to e^{i\theta(x)}\psi(x)$, so long as $\vec{A}$ makes a gauge transformation. Work out the transformation for $\vec{A}$ in terms of $\theta(x)$. [You only get half of the credit if you just remember the answer.] (10pts)

3) An electron moves in the x-y plane in the presence of a uniform magnetic field in the z-direction. Evaluate $[\Pi_x, \Pi_y]$ where $\Pi_x = p_x - eA_x/c$ and $\Pi_y = p_y - eA_y/c$. (10 pts)

4) Following 3), by comparing the hamiltonian $H = \Pi_x^2/2m + \Pi_y^2/2m$ and the commutation relation obtained in 3) with those of the one-dimensional oscillator, show the energy value is

$$E_n = \left(\frac{|eB|\hbar}{mc}\right)\left(n + \frac{1}{2}\right) \tag{4}$$

(10 pts)

Problem 2: Semiclassical Approximation (20 pts)

In the semiclassical approximation, the WKB wave function for a one-dimensional quantum mechanical system with hamiltonian, $H = p^2/2m + V(x)$, can be approximated as

$$\psi(x,t) \sim \frac{1}{[2m(E - V(x))]^{1/4}}\exp\left[\pm\frac{i}{\hbar}\int^x dx'\sqrt{2m[E - V(x')]}\right] \tag{5}$$

where $\sqrt{2m(E - V(x))}$ can be regarded as position-dependent classical momentum $p(x)$.

1) Calculate the probability density ρ and give a classical interpretation of its physical significance. (10 pts)

2) Use $J = (1/m)\mathrm{Re}[\psi^* \hat{p} \psi]$ to calculate the current density and give a classical interpretation. (10 pts)

Problem 3: Angular Momentum (40 pts)

Consider addition of orbital and spin-1 (s=1) angular momenta, $\vec{j} = \vec{\ell} + \vec{s}$. Assume the orbital motion is in $\ell = 1$ space.

1). Write down the basis of states which are eigenstates of $\vec{\ell}^2, \ell_z, \vec{s}^2, s_z$ for the tensor product space (let's call them m-states). (10 pts)

2). Find the total angular momentum state $|j = 2, m = 2\rangle$ in terms of the m-states. Using the lowering operator J_-, find state $|j = 2, m = 1\rangle$. Using orthogonal condition, find state $|j = 1, m = 1\rangle$. [Hint: $J_-|jm\rangle = \hbar\sqrt{(j + m)(j - m + 1)}|jm - 1\rangle$] (10 pts)

3). A spinor is a two-component complex vector ψ such that under space rotation $\vec{\theta}$, it transforms according to

$$\psi \to \exp\left(-i\frac{\vec{\theta} \cdot \vec{\sigma}}{2}\right)\psi \tag{6}$$

Assume the spin ($s = 1/2$) vector is in the $z - x$ plane, making an angle θ with respect to the z-axis. Work out the spinor by rotating the spinor in the z-direction (ψ_z = column vector (1,0)). (10 points)

4). An angular-momentum eigenstate $|j, m = m_{\mathrm{max}} = j\rangle$ is rotated by an infinitesimal angle ϵ about the y-axis. Without using the explicit form of the Wigner D-function, obtain an expression for the probability for the new rotated state to be in the original state up to terms of order ϵ^2. (10pts)