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Lecture 30, Parity, Monday, Nov. 21

Parity transformation refers to a reflection of the position vector in the 3-space,

$$\vec{r} \to -\vec{r} \ . \tag{1}$$

In the rectangular system of coordinates, this leads to $(x, y, z) \to (-x, -y, -z)$. In the spherical system of coordinates, the parity operation becomes

$$r \to r, \quad \theta \to \pi - \theta, \quad \phi \to \phi + \pi \ . \tag{2}$$

The mirror transformation $(x, y, z) \to (x, y, -z)$ is just the parity operation multiplied by a rotation.

In the quantum mechanical state space, let us denote the parity operation by a unitary operator π (please do not confuse it with 3.14159...). The parity operation on the quantum operators can be decided by physical considerations. For example, we must have

$$\pi \hat{\vec{r}} \pi^\dagger = -\hat{\vec{r}} \ . \tag{3}$$

Therefore the parity operator anti-commutes with the position

$$[\pi, \hat{\vec{r}}]_+ = 0 \ , \tag{4}$$

which means that one cannot diagonalize the position and parity operators simultaneously. Likewise, the momentum operator satisfies,

$$\pi \hat{\vec{p}} \pi^\dagger = -\hat{\vec{p}} \ . \tag{5}$$

Any vector which transforms like the position and momentum operators under parity is called a polar vector. On the other hand, the orbital angular momentum operator $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$ transforms as,

$$\pi \hat{\vec{L}} \pi^\dagger = \hat{\vec{L}} \ . \tag{6}$$

In other words, it commutes with the parity operator,

$$[\pi, \hat{\vec{L}}] = 0 \ . \tag{7}$$

And hence both can be diagonalized simultaneously.

It is clear that after two successive parity operations, one returns to the original state

$$\pi^2 = 1 \ . \tag{8}$$

Therefore, the eigenvalues of the parity operator are ± 1.

For a one-dimensional quantum mechanical system, the energy levels are non-degenerate. [This can be proved by inspecting the Schrodinger equation by assuming there are two solutions with the same energy.] For this reason, one can choose the wave functions to be real, because the complex conjugate wave function satisfy the same equation Further, if the hamiltonian is invariant under parity, which means that the potential is symmetric $V(x) = V(-x)$, the eigenfunctions must be eigenstates of parity. Let us consider a state with energy E,

$$H\psi(x) = E\psi(x) \ . \tag{9}$$

Applying the parity operator to the both sides of the equation, one find $\pi\psi(x)$ is also an eigenstate of H with the same eigenvalue. Since the energy levels are non-degenerate, we must have

$$\pi\psi(x) \sim \psi(x) \ . \tag{10}$$

Therefore, $\psi(x)$ is an eigenstate of parity. If the eigenvalue is $+1$, we call it parity-even, and if -1, we call it parity-odd. In other words,

$$\pi\psi_e(x) = \psi_e(x); \qquad \pi\psi_o(x) = -\psi_o(x) \ . \tag{11}$$

In a symmetric potential, the ground state of one-dim system is parity even. This is because the parity-even function has less wiggle and hence less kinetic energy. The first excited state is parity odd, with one node. The second excited state is parity even with two nodes. Clearly the wave functions with even numbers of nodes are parity-even, and with odd numbers of nodes are parity-odd. Since the number of nodes increases with the excitation energy, the eigenstates are alternatively even and odd in parity.