Lecture 2: Dirac Notation and Two-State Systems: Friday Sept. 2

Light polarization is an example of classical physics which uses the same kind of math structure as in quantum physics. Assuming you have already know the basics of quantum mechanics, let us summarize the main points of quantum principles here:

States and bras and kets

1) A state of a quantum mechanical system is *entirely* determined by a vector in a *complex* vector space (state space, called *Hilbert space* if it is infinite dimensional), just like the polarization of light. Such a vector is denoted by a ket $|\psi\rangle$, in Dirac notation. Dimensionality (finite or infinite) of the vector space depends on quantum systems.

2) Recall the conditions for a linear vector space: Two vectors can be added to obtain a third vector $c_1|\psi_1\rangle + c_2|\psi_2\rangle = |\psi\rangle$ (the principle of *superposition*, like in a wave).

3) $|\psi\rangle$ and $c|\psi\rangle$ represent the same state, and belong to the same "ray". Therefore, we also say that "a quantum state is represented by a ray."

4) For every ket $|\psi\rangle$, there is a corresponding bra $\langle\psi|$, which is called a dual vector. The dual of $c|\psi\rangle$ is $\langle\psi|c^*$, and the dual of $c_1|\psi_1\rangle + c_2|\psi_2\rangle$ is $c_1^*\langle\psi_1| + c_2^*\langle\psi_2|$.

5) The inner (scalar) product is defined between a bra and a ket, and the result is a complex number: $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* = c$. When switching the bra and ket, we get a complex-conjugate number.

6) $\langle\psi|\psi\rangle$ is real, we assume it is always positive definite. We call $\sqrt{\langle\psi|\psi\rangle}$ the length (norm) of the vector. Therefore, we can define a normalized vector $\langle\psi|\psi\rangle = 1$. We can assume all state vectors are normalized (with arbitrary phase of course).

7) Two vectors are orthogonal if their scalar product is zero $\langle\psi|\phi\rangle = 0$.

8) In the state space, one can define a set of orthonormal basis $|e_i\rangle$, $\langle e_i|e_j\rangle = \delta_{ij}$ and every vector in the space can be expanded in terms of the basis, $|\psi\rangle = \sum_i c_i|e_i\rangle$, where $c_i = \langle e_i|\psi\rangle$.

Observable and operators

1) All quantum mechanical observables are represented by linear, hermitian operators in the complex linear vector space. An operator O acting on a ket in the space yields another ket: $O|\psi\rangle = |\psi'\rangle$. Linearity means

$X(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle) = c_\alpha X|\alpha\rangle + c_\beta X|\beta\rangle$, and $(aX + bY)|\psi\rangle = aX|\psi\rangle + bY|\psi\rangle$. Operator product is noncommutative $XY \neq YX$.

2) The corresponding bra of $O|\psi\rangle$ is $\langle\psi|O^\dagger$, where $O^\dagger$ is called hermitian conjugation. A hermitian operator satisfies $O^\dagger = O$. The hermitian conjugation of XY is $Y^\dagger X^\dagger$.

3) Hermitian operators have real eigenvalues and the eigenstates with different eigenvalues are orthogonal.

4) If a hermitian operator has no degenerate eigenvalue, the eigenstates form an orthonormal basis in the state space.

5) Associativity of the product: $|\beta\rangle\langle\alpha||\gamma\rangle$ can be interpreted in two ways: a c-number times a ket, or an operator acting on a ket, both are the same. Therefore $|\psi\rangle\langle\phi|$ is an operator. Its hermitian conjugation is $|\phi\rangle\langle\psi|$. Like wise $\langle\phi|X|\psi\rangle$ can be interpreted in both ways. $\langle\beta|X|\alpha\rangle = \langle\alpha|X^\dagger|\beta\rangle^*$.

6) Closure relation: If we expand $|\psi\rangle = \sum_i c_i|e_i\rangle$, and use $c_i = \langle e_i|\psi\rangle$. Then we find $\sum_i |e_i\rangle\langle e_i| = 1$.

7) Projection operator: $P = |\psi\rangle\langle\psi|$ is a projection operator in the sense that when it acts on any state, it projects along the direction of $|\psi\rangle$. Define $P_i = \langle e_i\rangle\langle e_i|$, then $\sum_i P_i = 1$.

8) Matrix representation: A state ket, once expanded in a basis, can be represented by a column matrix. The same state, once written as a bra, can be represented by a row matrix with complex-conjugated numbers. An operator can be expanded in terms of $|e_i\rangle\langle e_j|$. The coefficient of the expansion is $\langle e_i|O|e_j\rangle$ and can be arranged in the form of a square matrix. All expressions in terms of bras, kets, and operators can be converted into matrix expressions.

Example

The simplest quantum mechanical systems have their state spaces as a 2D complex vector space. Examples include spin-1/2 particles such as the electron or proton, neutrino flavor oscillations, kaon decay, NH_3 molecules. [In fact, light polarization is a phenomenon resulted from quantum mechanics of photon polarization!] In the 2D space, we can take a basis $|+\rangle$ and $|-\rangle$. Then all operators can be taken as 2×2 hermitian matrices. Each of these can be written as a linear superposition of the following four hermitian matrices,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (22)$$

where σ_i are called Pauli matrices. Thus we write,

$$O = a_0 I + \sum_i a_i \sigma_i \tag{23}$$

where a_0 and a_i are real numbers. Therefore, independent of whether we are dealing with a spin system or not, Pauli matrices are very useful for doing calculations in 2D vector spaces.

For a spin-1/2 particle, the spin angular momentum operators can be identified with Pauli matrices. In particular,

$$S_i = \frac{\hbar}{2}\sigma_i \tag{24}$$

where S_i satisfy the angular momentum commutation relation,

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k \tag{25}$$

Clearly $|+$ and $|-\rangle$ are eigenstates of S_z with eigenvalues $\hbar/2$ and $-\hbar/2$, respectively.

The magnetic moment of a spin-1/2 particle is proportional to its angular momentum

$$\vec{\mu} = g\mu_B\vec{S} \tag{26}$$

where $\mu_B = |e|\hbar/2mc$ and g is called g-factor. For the electron it close to -2, where negative sign arises because the electron has negative charge (spin and magnetic moment have different direction). The interaction energy in a magnetic field is,

$$H = -\vec{\mu}\cdot\vec{B}\ , \tag{27}$$

which H is an operator in the state space.

The eigenstates of S_x are $|\pm\rangle_x = (|+\rangle \pm |-\rangle)/\sqrt{2}$, with eigenvalues $\pm\hbar/2$. The eigenstates of S_y are $|\pm\rangle_y = (|+\rangle \pm i|-\rangle)/\sqrt{2}$, with the same eigenvalues. The eigenstates of

$$\vec{n}\cdot\vec{S} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \tag{28}$$

where $\vec{n} = (\cos\phi\sin\theta, \sin\theta\sin\phi, \cos\theta)$ can be found to be

$$\begin{aligned} |+\rangle_n &= \cos\frac{\theta}{2}e^{-i\phi/2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi/2}|-\rangle \\ |-\rangle_n &= -\sin\frac{\theta}{2}e^{i\phi/2}|+\rangle + \cos\frac{\theta}{2}e^{-i\phi/2}|-\rangle \end{aligned} \tag{29}$$