Phys. 622: Homework Set II

1. (JJS 1.23) Consider a three-dimensional ket space. If a certain set of orthonormal kets, say $|1\rangle$, $|2\rangle$, $|3\rangle$, are used as the basis kets, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}; \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \tag{1}$$

with a and b both real.

a. Obviously A exhibits a degenerate spectrum. Does B also exhibit a degenerate spectrum?

b. show that A and B commute.

c. Find a new set of orthonormal kets which are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

2 (JJS 1.13). A beam of spin 1/2 atoms goes through a series of Stern-Gerlach type measurements as follows:

a. The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.

b. The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where s_n is the eigenvalue of the operator $\vec{S} \cdot \hat{n}$, with $\hat{n}$ making an angle β in the xz-plane with respect to the z-axis.

c. The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement of normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam?

3. Consider a physical system whose state space, which is three-dimensional, is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$. In this basis, the Hamiltonian operator H of the system and the two observables A and B are written

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \quad A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad B = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{2}$$

The physical system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle \tag{3}$$

a. Suppose the energy of the system is measured, what values can be found, and with what probabilities? Calculate, for the system in the state the mean value $\langle H\rangle$ and the root-mean-square deviation ΔH.

b. Suppose instead one measures A, what results can be found and with what probabilities? what is the state vector immediately after the measurement?

c. What is the root mean-square deviation ΔB?

4 (JJS 1.20). Find the linear combination of $+\rangle$ and $|-\rangle$ kets that maximizes the uncertainty product $\langle(\Delta S_x)^2(\Delta S_y)^2\rangle$. Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

5. For the two spin state

$$\begin{aligned} |\Psi\rangle &= \frac{1}{\sqrt{2}}|+\rangle_A\left(\frac{1}{2}|+\rangle_B + \frac{\sqrt{3}}{2}|-\rangle_B\right) \\ &+ \frac{1}{\sqrt{2}}|-\rangle_A\left(\frac{\sqrt{3}}{2}|+\rangle_B + \frac{1}{2}|-\rangle_B\right) \end{aligned} \tag{4}$$

a. Compute $\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$ and $\rho_B = \mathrm{Tr}_A(|\Psi\rangle\langle\Psi|)$.

b. Find the Schmidt decomposition of $|\Psi\rangle$ (optional).

6. Consider a density matrix for two spins

$$\rho = \frac{1}{8}I + \frac{1}{2}|\psi^-\rangle\langle\psi^-| \tag{5}$$

where I is a 4×4 unit matrix, and

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle) \tag{6}$$

Suppose we measure the first spin along the $\hat{n}$ axis and the second spin along the $\hat{m}$ axis, where $\hat{n}\cdot\hat{m} = \cos\theta$. What is the probability that both spins are up along their respective axis?