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Homework 2

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Problem 1

(a)

Using the matrix representation of the field tensor, we can write (taking $c = 1$)

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= \sum_{i\neq 0} F_{0i}F^{0i} + \sum_{i\neq 1} F_{1i}F^{1i} + \sum_{i\neq 2} F_{2i}F^{2i} + \sum_{i\neq 3} F_{3i}F^{3i} \\ &= [-E_x^2 - E_y^2 - E_z^2] + [-E_x^2 + B_z^2 + B_y^2] + [-E_y^2 + B_z^2 + B_x^2] + [-E_z^2 + B_y^2 + B_x^2] \\ &= 2(\boldsymbol{B}^2 - \boldsymbol{E}^2) \end{aligned} \tag{1}$$

Now, $\epsilon^{\mu\nu\lambda\rho}$ is antisymmetric in any two indices and is $+1$ or -1 depending on whether $\mu\nu\lambda\rho$ is an even or odd permutation of 0123. So, we need only consider the case $\mu \neq \nu \neq \lambda \neq \rho$. If $\mu\nu = 01$ or 10, then $\lambda\rho = 23$ or 32. Each of the four terms equals $-E_xB_x$. Also, if $\mu\nu = 23$ or 32 and $\lambda\rho = 01$ or 10, we get four terms each equal to $-E_xB_x$. Therefore, there are 8 occurrences of $-E_xB_x$ in the sum $\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$. By a similar reasoning, there are 8 occurrences each of $-E_yB_y$ and $-E_zB_z$. Therefore,

$$\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = -8\boldsymbol{E}\cdot\boldsymbol{B} \tag{2}$$

Here we have adopted the convention $\epsilon^{0123} = +1$.

(b)

The quantity $\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = -8\boldsymbol{E}\cdot\boldsymbol{B}$ is a Lorentz scalar, and hence is invariant under Lorentz transformations. Therefore, if $(\boldsymbol{E}, \boldsymbol{B})$ and $(\boldsymbol{E}', \boldsymbol{B}')$ are the electric and magnetic fields in two frames of reference connected to each other via a Lorentz transformation, then

$$\boldsymbol{E}\cdot\boldsymbol{B} = \boldsymbol{E}'\cdot\boldsymbol{B}' \tag{3}$$

So, if $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal, $\boldsymbol{E}\cdot\boldsymbol{B} = 0$, and hence $\boldsymbol{E}'\cdot\boldsymbol{B}' = 0$ which implies that $\boldsymbol{E}'$ and $\boldsymbol{B}'$ are also orthogonal.

(c)

The quantity, $F_{\mu\nu}F^{\mu\nu} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2)$ is invariant under Lorentz transformations. So, $\boldsymbol{B}^2 - \boldsymbol{E}^2 = \boldsymbol{B'}^2 - \boldsymbol{E'}^2$. Therefore, if $|\boldsymbol{B}| = |\boldsymbol{E}|$ then $|\boldsymbol{B}'| = |\boldsymbol{E}'|$.

(d)

From the previous part, $\boldsymbol{B}^2 - \boldsymbol{E}^2 = \boldsymbol{B'}^2 - \boldsymbol{E'}^2$. Therefore, if $|\boldsymbol{E}|^2 > |\boldsymbol{B}|^2$ then $|\boldsymbol{E}'|^2 > |\boldsymbol{B}'|^2$ and hence $|\boldsymbol{E}'| > |\boldsymbol{B}'|$.

Problem 2

(a)

Assume that the z-axis is along the length of the wire. In the rest frame K' of the wire, consider a cylinder of radius ρ and length L. The charged 'enclosed' by the volume of the cylinder is $Q_{enc} = q_0 L$. By symmetry, the electric field points radially outwards from the wire, along the direction $\hat{e}_\rho$. Applying Gauss's Law,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{s} = 4\pi Q_{enc} \qquad \text{(Gaussian Units)} \tag{4}$$

we get

$$E \times (2\pi\rho L) = 4\pi q_0 \times L$$

So, the electric field in the rest frame K' of the wire is

$$\boxed{\boldsymbol{E}_{K'} = \frac{2q_0}{\rho}\hat{e}_\rho \qquad \text{(Rest Frame; Gaussian Units)}} \tag{5}$$

In the rest frame of the wire, the charges are static. Therefore the current, and hence the magnetic field are both zero.

$$\boxed{\boldsymbol{B}_{K'} = 0 \qquad \text{(Rest Frame)}} \tag{6}$$

Lorentz Transformation of the Fields

Let $F^{\mu\nu}$ denote the second rank antisymmetric field tensor in the lab frame. The rest frame of the wire, K' is moving with a velocity $v\hat{z}$ with respect to the lab frame K. We know the components of $F^{\mu\nu\prime}$, i.e. the field tensor in the rest frame, and want to determine the components of $F^{\mu\nu}$.

Now, $F^{\mu\nu}$ transforms like a second rank antisymmetric field tensor under a Lorentz transformation. In this case, the Lorentz transformation is a Lorentz boost along the z-axis (x^3 direction). Under the boost, x^1 and x^2 remain invariant. F^{01} and F^{02} transform like x^0. Let $\gamma = 1/\sqrt{1 - v^2/c^2}$. So,

$$F^{01} = \gamma\left(F^{01\prime} - \frac{(-v)}{c^2}F^{31\prime}\right) \implies \boxed{E_x = \gamma\left(E'_x - \frac{v}{c}B'_y\right)} \tag{7}$$

$$F^{02} = \gamma\left(F^{02\prime} - \frac{(-v)}{c^2}F^{32\prime}\right) \implies \boxed{E_y = \gamma\left(E'_y + \frac{v}{c}B'_x\right)} \tag{8}$$

The components $F^{03} = -F^{30}$ and $F^{33} = F^{00} = 0$ are invariant under the rotation in x^0x^3 space, as they are components of a rank-2 antisymmetric tensor in a 2-dimensional space. Therefore,

$$\boxed{E'_z = E_z} \tag{9}$$

Now, to the magnetic fields: F^{12} remains invariant as x^1, x^2 are invariant, F^{13} and F^{23} transform like x^3. Therefore,

$$F^{12} = F^{12\prime} \implies \boxed{B_z = B_z'} \tag{10}$$

$$F^{13} = \gamma\left(F^{13\prime} - (-v)F^{10\prime}\right) \implies \boxed{B_y = \gamma\left(B_y' + \frac{v}{c}E_x'\right)} \tag{11}$$

$$F^{23} = \gamma\left(F^{23\prime} - (-v)F^{20\prime}\right) \implies \boxed{B_x = \gamma\left(B_x' - \frac{v}{c}E_y'\right)} \tag{12}$$

In this problem, $E_z' = 0$, $B_x' = B_y' = B_z' = 0$. So,

$$E_x = \gamma E_{x'} \tag{13}$$
$$E_y = \gamma E_{y'} \tag{14}$$
$$E_z = 0 \tag{15}$$
$$B_x = -\gamma\frac{v}{c}E_y' \tag{16}$$
$$B_y = \gamma\frac{v}{c}E_x' \tag{17}$$
$$B_z = 0 \tag{18}$$

So, the lab frame electric and magnetic fields are

$$\boldsymbol{E}_K = \frac{2\gamma q_0}{\rho}\hat{e}_\rho \qquad \text{(Lab Frame; Gaussian Units)} \tag{19}$$
$$\boldsymbol{B}_K = \frac{2\gamma q_0 v}{c\rho}\hat{e}_\phi \qquad \text{(Lab Frame; Gaussian Units)} \tag{20}$$

where we have used

$$\hat{e}_\rho = \cos\phi\hat{e}_x + \sin\phi\hat{e}_y \tag{21}$$
$$\hat{e}_\phi = -\sin\phi\hat{e}_x + \cos\phi\hat{e}_y \tag{22}$$

for conversion to cylindrical coordinates ($\hat{e}_x$, $\hat{e}_y$, $\hat{e}_\rho$, $\hat{e}_\theta$ all lie in the plane perpendicular to the direction of the boost, and hence remain invariant).

(b)

The charge and current densities in the rest frame of the wire are $\boxed{\lambda_{wire} = q_0}$ (statcoulomb per unit length) and $\boxed{\mathcal{J}_{wire} = 0}$ (statampere per unit length) respectively, as the wire is static in this frame and there is no current. In the lab frame, the system corresponds to a charged wire moving at a constant speed. Comparing Eqn. (19) to the expression for the electric field produced by an infinite uniformly charged wire, the charge density is found to be

$$\boxed{\lambda_{lab} = \gamma\lambda_{wire} = \frac{q_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{Charge density; Lab frame}} \tag{23}$$

And comparing Eqn. (20) to the expression for the magnetic field produced by a uniform current carrying wire ($B = 2I/c\rho$), the current density is found to be

$$\boxed{\mathcal{J}_{lab} = \frac{q_0 v \delta(\rho)}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{Current density; Lab frame}} \tag{24}$$

Here $\delta(\rho)$ is the Dirac delta 'function', which provides an additional $(\text{length})^{-1}$ dimension so that j_{lab} has the dimensions of a current density, and also emphasizes the fact that the current is concentrated along $\rho = 0$, which in cylindrical coordinates corresponds to the z axis.

An **alternate** method for obtaining this result is as follows: consider a segment of length L of the wire in its rest frame. The charge in this segment is $q_0 \times L$. Due to length contraction, this segment has a length L/γ in the lab frame. Applying charge conservation $\lambda_{lab}\frac{L}{\gamma} = q_0 \times L$ which gives $\lambda_{lab} = \gamma q_0$. The result is physically consistent: the moving wire appears contracted and hence the same charge is now distributed over a shorter length. (The argument is being applied to any finite segment of the infinite wire. The total charge is still infinite.) Also, the current density in the rest frame is zero, $\mathcal{J}_{wire} = 0$. Now, $j^\mu = (\lambda, \boldsymbol{\mathcal{J}} L/c)$ forms a four vector, where L is the length of a segment of the wire along the z-axis. Using Lorentz transformation, we get

$$\mathcal{J}_{lab}\frac{L}{c} = \gamma\left(\mathcal{J}_{wire}\frac{L}{c} + \frac{v}{c}\lambda_{wire}\right) = \gamma\frac{v}{c}\lambda_{wire} \tag{25}$$

$$\implies \mathcal{J}_{lab} = \frac{\gamma v \lambda_{wire}}{L} \tag{26}$$

Here $1/L$ plays the role of $\delta(\rho)$ in Eqn. (24).

(c)

By symmetry, the Electric field in the lab frame is along $\hat{e}_\rho$ and is given by

$$\boldsymbol{E}_{lab} = \frac{2\lambda_{lab}}{\rho}\hat{e}_\rho \tag{27}$$

$$= \frac{2\gamma q_0}{\rho}\hat{e}_\rho \tag{28}$$

whereas the Magnetic field in the lab frame is along $\hat{e}_\phi$ and is given by

$$\boldsymbol{B}_{lab} = \frac{2I}{c\rho}\hat{e}_\rho \tag{29}$$

$$= \frac{2\mathcal{J}_{lab}L}{c\rho}\hat{e}_\rho \qquad \text{(for a small segment of length } L\text{)}$$

$$= \frac{2\gamma q_0 v}{c\rho}\hat{e}_\rho \qquad \text{(using 26)} \tag{30}$$

The results agree with those obtained in part (a). Note that in fixing the directions of $\boldsymbol{E}$ and $\boldsymbol{B}$ in the lab frame, we have referred to "symmetry". The directions can be more rigorously established by using parts (a) and (d) of problem 1, along with the fact that the unit vectors, which lie in a plane orthogonal to the direction of spatial boost, are invariant.

Problem 3

Suppose F' is a frame in which $\boldsymbol{E}'$ and $\boldsymbol{B}'$ are parallel. Then, further boosting along the common direction of $\boldsymbol{E}'$ and $\boldsymbol{B}'$, we can obtain another frame F'' in which the fields are parallel to each other. This shows that F' is not unique. We can find one such frame by boosting along the z-axis of the lab frame, to a frame F'. Using the transformations employed in Problem 2, the electric field components in F' are found to be

$$E'_x = \gamma\left(E_x - \frac{v}{c}B_y\right) \tag{31}$$

$$E'_y = \gamma\left(E_y + \frac{v}{c}B_x\right) \tag{32}$$

$$E'_z = E_z \tag{33}$$

whereas the magnetic field components are found to be

$$B'_x = \gamma\left(B_x + \frac{v}{c}E_y\right) \tag{34}$$

$$B'_y = \gamma\left(B_y - \frac{v}{c}E_x\right) \tag{35}$$

$$B'_z = B_z \tag{36}$$

The condition for $\boldsymbol{E}'$ and $\boldsymbol{B}'$ to be parallel to each other is $\boldsymbol{E}'\times\boldsymbol{B}' = 0$. This can be further simplified by observing that $E_z = 0$ and $B_z = 0$. In particular we need only consider $(\boldsymbol{E}'\times\boldsymbol{B}')_z = E'_xB'_y - E'_yB'_x = 0$. This gives

$$(E_xB_y - E_yB_x) - \frac{v}{c}(E_x^2 + E_y^2) - \frac{v}{c}(B_x^2 + B_y^2) + \frac{v^2}{c^2}(E_xB_y - B_xE_y) = 0 \tag{37}$$

$$\implies (\boldsymbol{E}\times\boldsymbol{B})_z - \frac{v}{c}(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2) + \frac{v^2}{c^2}(\boldsymbol{E}\times\boldsymbol{B})_z = 0 \tag{38}$$

Eqn. (38) is a quadratic in v/c with two roots,

$$\frac{v}{c} = \frac{(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2) \pm \sqrt{(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2)^2 - 4(\boldsymbol{E}\times\boldsymbol{B})_z^2}}{2(\boldsymbol{E}\times\boldsymbol{B})_z} \tag{39}$$

As $v < c$, the only physically meaningful root corresponds to the taking the $-$ sign in Eqn. (39), so that

$$\boxed{\frac{v}{c} = \frac{(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2) - \sqrt{(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2)^2 - 4(\boldsymbol{E}\times\boldsymbol{B})_z^2}}{2(\boldsymbol{E}\times\boldsymbol{B})_z}} \tag{40}$$

Thus, one of the infinitely many frames in which $\boldsymbol{E}||\boldsymbol{B}$ can be obtained by boosting along the z-axis (i.e. along $\boldsymbol{E}\times\boldsymbol{B}$) by a speed given by Eqn. (40). For this problem,

$$\boldsymbol{E} = E_0\hat{x} \tag{41}$$

$$\boldsymbol{B} = 2E_0(\cos\theta\hat{x} + \sin\theta\hat{y}) \tag{42}$$

so that

$$\boxed{\frac{v}{c} = \frac{1 - \sqrt{1 - \frac{16}{25}\sin^2\theta}}{\frac{4}{5}\sin\theta}} \tag{43}$$

For $\theta << 1$,

$$\begin{aligned}\frac{v}{c} &\approx \frac{1-\sqrt{1-\frac{16}{25}\theta^2}}{\frac{4}{5}\theta} \\ &\approx \frac{1-\left(1-\frac{8}{25}\theta^2\right)}{\frac{4}{5}\theta} \\ &\approx \frac{2}{5}\theta \end{aligned} \tag{44}$$

whereas for $\theta \to \pi/2$,

$$\frac{v}{c} \to \frac{1-\sqrt{1-\frac{16}{25}}}{\frac{4}{5}} = \frac{1}{2} \tag{45}$$

So,

$$\boxed{\begin{aligned}\theta << 1 \implies \frac{v}{c} \approx \frac{2}{5}\theta \quad (46)\\ \theta \to \frac{\pi}{2} \implies \frac{v}{c} \to \frac{1}{2} \quad (47)\end{aligned}}$$

$\boxed{\theta << 1 \textbf{ regime}}$

Plugging $\frac{v}{c} = 2\theta/5$ ($\gamma = \frac{1}{\sqrt{1-\frac{4}{5}\theta^2}}$) in the expressions for the electric and magnetic field, we get

$$E'_x = \gamma\left(E_x - \frac{v}{c}B_y\right) \approx \frac{E_0\left(1-\frac{4}{5}\theta\right)}{\sqrt{1-\frac{4}{5}\theta^2}} \tag{48}$$

$$E'_y = \gamma\left(E_y + \frac{v}{c}B_x\right) \approx \frac{\frac{4E_0\theta}{5}}{\sqrt{1-\frac{4}{5}\theta^2}} \tag{49}$$

$$E'_z = E_z = 0 \tag{50}$$

$$B'_x = \gamma\left(B_x + \frac{v}{c}E_y\right) \approx \frac{2E_0}{\sqrt{1-\frac{4}{5}\theta^2}} \tag{51}$$

$$B'_y = \gamma\left(B_y - \frac{v}{c}E_x\right) \approx \frac{2E_0\theta - \frac{2E_0\theta}{5}}{\sqrt{1-\frac{4}{5}\theta^2}} = \frac{\frac{8E_0\theta}{5}}{\sqrt{1-\frac{4}{5}\theta^2}} \tag{52}$$

$$B'_z = B_z = 0 \tag{53}$$

If $\theta = 0$, $E'_x = E_0$, $E'_y = E'_z = 0$, $B'_x = 2E_0$, $B'_y = B'_z = 0$, which is sensible as both the fields are parallel for $\theta = 0$.

$\theta \to \dfrac{\pi}{2}$ **regime**

For $\theta \to \pi/2$, $\frac{v}{c} \to \frac{1}{2}$, $\gamma \to \frac{2}{\sqrt{3}}$. Plugging these into the field equations, we get

$$E'_x = \gamma\left(E_x - \frac{v}{c}B_y\right) \to 0 \tag{54}$$
$$E'_y = \gamma\left(E_y + \frac{v}{c}B_x\right) \to 0 \tag{55}$$
$$E'_z = E_z = 0 \tag{56}$$
$$B'_x = \gamma\left(B_x + \frac{v}{c}E_y\right) \to 0 \tag{57}$$
$$B'_y = \gamma\left(B_y - \frac{v}{c}E_x\right) \to \frac{2}{\sqrt{3}}\frac{3}{2}E_0 = \sqrt{3}E_0 \tag{58}$$
$$B'_z = B_z = 0 \tag{59}$$

Problem 4

The trajectory of the particle will be circular. The equation of motion is

$$\frac{d\boldsymbol{p}}{dt} = q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \tag{60}$$

Suppose $\boldsymbol{B} = B\hat{z}$. Now, the energy and momentum are

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{61}$$
$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{62}$$

So,

$$\boldsymbol{p} = \frac{\mathcal{E}\boldsymbol{v}}{c^2} \tag{63}$$

As the energy $\mathcal{E}$ is constant in the magnetic field, we can write $\dot{\boldsymbol{p}} = \frac{\mathcal{E}}{c^2}\dot{\boldsymbol{v}}$, so the equation of motion becomes

$$\frac{\mathcal{E}}{c^2}\frac{d\boldsymbol{v}}{dt} = q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \tag{64}$$

Define $\omega = \frac{qBc}{\mathcal{E}}$. Then,

$$\dot{v}_x = \omega v_y \tag{65}$$
$$\dot{v}_y = -\omega v_x \tag{66}$$
$$\dot{v}_z = 0 \tag{67}$$

As the initial velocity is directed perpendicular to the field, $v_z = 0$. The first two differential equations are coupled, and can be decoupled by differentiating once with respect to time, yielding

$$\ddot{v}_x + \omega^2 v_x = 0 \tag{68}$$
$$\ddot{v}_y + \omega^2 v_y = 0 \tag{69}$$

So,

$$v_x(t) = A\cos(\omega t + \alpha) \tag{70}$$
$$v_y(t) = -A\sin(\omega t + \alpha) \tag{71}$$

Integrating again, we get

$$x(t) = x_0 + \frac{A}{\omega}\sin(\omega t + \alpha) \tag{72}$$
$$y(t) = y_0 + \frac{A}{\omega}\cos(\omega t + \alpha) \tag{73}$$

So, the trajectory is a circle, as claimed above, with a **radius** given by

$$\boxed{r = \frac{A}{\omega} = \frac{A\mathcal{E}}{qBc}} \tag{74}$$

Here $A = p_t c^2/\mathcal{E}$ (from Eqn. (63)) where p_t is the transverse momentum, that is, the projection of $\boldsymbol{p}$ on the plane of the circle. So,

$$\boxed{r = \frac{p_t c}{qB} \qquad \text{(Gaussian Units)}} \tag{75}$$

As $p_z = 0$, therefore $p_t = |\boldsymbol{p}|$, the magnitude of the total momentum of the particle. Therefore,

$$\boxed{r = \frac{|\boldsymbol{p}|c}{qB} = \frac{\sqrt{\left(\frac{\mathcal{E}}{c}\right)^2 - m^2c^2}\,c}{qB} \qquad \text{(Gaussian Units)}} \tag{76}$$

where $m = 0.511MeV$ is the mass of the electron.

(a) $\mathcal{E} = mc^2 + 1000eV$

For convenience, we work in SI units, where $r = \frac{|\boldsymbol{p}|}{qB}$. In this case, $\mathcal{E} = 8.192 \times 10^{-14}\,J$, $m = 9.1 \times 10^{-31}kg$, so $|\boldsymbol{p}|c = \sqrt{E^2 - m^2c^4}$ so $|\boldsymbol{p}|c = 5.1175 \times 10^{-15}\,J$. Substituting these into the expression for the radius, we get

$$\boxed{r = 1.06615 \times 10^{-5}\,m \qquad \text{(KE = 1000 eV)}} \tag{77}$$

(b) $\mathcal{E} = mc^2 + 100GeV$

In this case, $\mathcal{E} = 1.60001 \times 10^{-8}\,J$, and $|\boldsymbol{p}|c = 1.60001 \times 10^{-8}\,J$. So, the radius of curvature is

$$\boxed{r = 33.3335\,m \qquad \text{(KE = 100 GeV)}} \tag{78}$$

Problem 5

Consider the frame in which the charged particle is initially at rest. The total four-momentum before emission of the photon is

$$p^\mu_{TOT} = (m, 0, 0, 0) \tag{79}$$

where we have taken $c = 1$. Also $m \neq 0$. Assuming that the photon is emitted along the x-axis in this frame, after emission of the photon, the four momentum of the charged particle is

$$p^\mu_1 = (E_1, p, 0, 0) \tag{80}$$

where E_1 is the energy of the charged particle after photon emission. We can assume $p \geq 0$ without loss of generality. By momentum conservation, the four momentum of the photon is

$$p^\mu_2 = (p, -p, 0, 0) \tag{81}$$

As $p^\mu_{TOT} = p^\mu_1 + p^\mu_2$, we must have

$$E_1 + p = m \tag{82}$$

Also, $E_1^2 = p^2 + m^2$. So,

$$\begin{aligned} E_1 &= m - p \\ \implies E_1^2 &= m^2 + p^2 - 2mp \\ \implies p^2 + m^2 &= m^2 + p^2 - 2mp \\ \implies 2pm &= 0 \\ \implies p &= 0 \text{ or } m = 0 \end{aligned} \tag{83}$$

As m is given to be non-vanishing, it follows that $p = 0$, which implies that no photon is emitted. This shows that a charged particle with a non-vanishing mass cannot emit a photon *and* stay with the same mass.

Problem 6

In the lab frame, the four momenta before collision are

$$(p_1^{LAB})^\mu = (E, p, 0, 0) \tag{84}$$

$$(p_2^{LAB})^\mu = (m, 0, 0, 0) \tag{85}$$

where we have used $c = 1$. Also, $E^2 = p^2 + m^2$. Suppose we boost to a frame moving with a velocity v. Let $\gamma = 1/\sqrt{1 - v^2}$. Then, the components of the four momenta transform as

$$E_1^{CM} = \gamma(E - vp) \tag{86}$$

$$E_2^{CM} = \gamma(m - v \times 0) = \gamma m \tag{87}$$

$$p_{1,x}^{CM} = \gamma(p - vE) \tag{88}$$

$$p_{2,x}^{CM} = \gamma(0 - vm) = -\gamma v m \tag{89}$$

In the CM frame, the 3-momenta sum to zero, i.e. $p_{1,x}^{CM} + p_{2,x}^{CM} = 0$. So,

$$v = \frac{p}{E+m} \tag{90}$$

Substituting this into (86) and (87) we get

$$\begin{aligned}
E_1^{CM} &= \gamma\left(E - \frac{p^2}{E+m}\right) \\
&= \gamma\left(\frac{E^2 + Em - p^2}{E+m}\right) \\
&= \gamma\left(\frac{m(E+m)}{E+m}\right) \\
&= \gamma m \qquad (91) \\
E_2^{CM} &= \gamma m \qquad (92)
\end{aligned}$$

Also, $\gamma = \frac{1}{\sqrt{1-v^2}} = \sqrt{\frac{m+E}{2m}}$. So the total energy in the CM frame is

$$\boxed{E_{CM}^{TOT} = \sqrt{2m(E+m)}} \tag{93}$$

For $E >> m$,

$$E_{CM}^{TOT} \approx \sqrt{2mE} = \sqrt{\frac{2m}{E}}E \tag{94}$$

which shows that E_{CM}^{TOT} is a small fraction of the total energy E.

If instead, we arrange a **head-on collision** of two particles of mass m and energy E in the lab frame, then the "lab" frame and "CM" frame are identical, and the total energy in this frame is **<u>2E</u>** (equal to the sum of the total energies of the incident particles). This is clearly larger than $\sqrt{\frac{m}{2E}}2E$ in the $E >> m$ regime.

In fact the total energy in this case is larger than the total energy in the CM frame in the fixed target frame. This is proved below.

$$\begin{aligned}
& 2E > \sqrt{2m(E+m)} && (95) \\
\iff\ & 4E^2 > 2m(E+m) \\
\iff\ & 2E^2 - mE + m^2 > 0 \\
\iff\ & (-m)^2 - 4(2)(m^2) < 0 \qquad \text{(discriminant } < 0 \text{ for quadratic form to be } > 0) \\
\iff\ & -7m^2 < 0, \text{ which is an identity.} && (96)
\end{aligned}$$

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