

Solutions to Exam 1

Physics 604

Oct. 23, 2003

This CLOSED BOOK exam. Three problems are to be completed in 75 minutes and carry equal weight. You must explain your reasoning to receive full credit.

▼ Problem 1

Suppose that $f_2[z]$ is analytic at z and that $f_1[w]$ is analytic at $w = f_2[z]$. Show that $f[z] = f_1[f_2[z]]$ is analytic at z .

solution

Let

$$\begin{aligned}z &= x + i y \\w = f_2[z] &= u[x, y] + i v[x, y] \\f_1[w] &= p[u, v] + i q[u, v]\end{aligned}$$

such that

$$f[z] = f_1[w] = f_1[f_2[z]] = p[u[x, y], v[x, y]] + i q[u[x, y], v[x, y]]$$

Analyticity of $f_2[z]$ and $f_1[w]$ requires that continuous first partial derivatives of the component functions exist at z for (u, v) and at w for (p, q) and satisfy the Cauchy-Riemann equations:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, & \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ \frac{\partial p}{\partial u} &= \frac{\partial q}{\partial v}, & \frac{\partial p}{\partial v} &= -\frac{\partial q}{\partial u}\end{aligned}$$

Use of the chain rule for these real-valued functions of real variables

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial p}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial p}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial q}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial q}{\partial u} \frac{\partial u}{\partial y} = \frac{\partial q}{\partial y} \\ \frac{\partial p}{\partial y} &= \frac{\partial p}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial p}{\partial v} \frac{\partial v}{\partial y} = -\frac{\partial q}{\partial v} \frac{\partial v}{\partial x} - \frac{\partial q}{\partial u} \frac{\partial u}{\partial x} = -\frac{\partial q}{\partial x}\end{aligned}$$

then results in the Cauchy-Riemann equations for $f[z]$ and demonstrates the analyticity of composition of analytic functions.

▼ Problem 2

Use contour integration to evaluate

$$\int_0^\pi \sin[\theta]^{2n} d\theta$$

for nonnegative integer n .

solution

Using symmetry and

$$z = e^{i\theta}, \quad d\theta = \frac{dz}{iz}, \quad \text{Sin}[\theta] = \frac{z - z^{-1}}{2i}$$

we obtain

$$I = \int_0^\pi \text{Sin}[\theta]^{2n} d\theta = \frac{1}{2} \int_0^{2\pi} \text{Sin}[\theta]^{2n} d\theta = \frac{1}{2i} \oint_C \left(\frac{z - z^{-1}}{2i} \right)^{2n} \frac{dz}{z} = \frac{(-)^n}{2^{2n+1}i} \oint_C \frac{(z^2 - 1)^{2n}}{z^{2n+1}} dz$$

where C is the unit circle. The integrand has a pole at the origin of order $2n + 1$. Using the binomial theorem

$$(z^2 - 1)^{2n} = \sum_{m=0}^{2n} \binom{2n}{m} (-)^m (z^2)^{2n-m}$$

we identify the residue as the coefficient of z^{2n} , which is the term with $m = n$, and obtain

$$I = 2\pi i \frac{(-)^n}{2^{2n+1}i} \binom{2n}{n} (-)^n = \frac{\pi}{4^n} \frac{(2n)!}{(n!)^2}$$

▼ Problem 3

Evaluate the principal-value integral

$$\mathcal{P} \int_0^\infty \frac{x^{1/4}}{x^2 - x - 2} dx$$

Sketch your contour and justify clearly any portions that are neglected.

solution

Let

$$f[z] = \frac{z^{1/4}}{(z+1)(z-2)}, \quad z^{1/4} = |z|^{1/4} \text{Exp}\left[\frac{i}{4} \text{Arg}[z]\right]$$

where $0 \leq \text{Arg}[z] < 2\pi$ is cut along the positive real axis. There is a simple pole at $z = -1$ on the negative real axis and a pole at $z = 2$ on the cut. Applying the residue theorem to the contour sketched below we find

$$\oint_C \frac{z^{1/4}}{(z+1)(z-2)} dz = 2\pi i \frac{e^{i\pi/4}}{-3}$$

using the residue of the pole at $z = -1$. Note that

$$z = R e^{i\theta} \implies \left| f[z] \frac{dz}{d\theta} \right| \approx R^{-3/4} \rightarrow 0$$

is negligible on a great circle with $R \rightarrow \infty$ and

$$z = \varepsilon e^{i\theta} \implies \left| f[z] \frac{dz}{d\theta} \right| \approx \frac{1}{2} \varepsilon^{5/4} \rightarrow 0$$

is negligible on an infinitesimal circle around the origin. Therefore, we can divide the contours along the real axis into principal value and pole contributions

$$\oint_C \frac{z^{1/4}}{(z+1)(z-2)} dz = (1-i) \mathcal{P} \int_0^\infty \frac{x^{1/4}}{x^2-x-2} dx - i\pi \frac{2^{1/4}}{3} (1+i)$$

where the phases on opposite sides differ by 2π . Combining these results and using $(1+i)/(1-i) = i$, we finally obtain

$$\mathcal{P} \int_0^\infty \frac{x^{1/4}}{x^2-x-2} dx = \frac{\pi}{3} (2^{1/2} - 2^{1/4})$$

