

Physics 603 D. W. Greenberg
 Solutions to homework due 5/6/03

8.10. This problem is similar to Problem 7.14 of the Bose gas and can be done the same way.

Parts (i) and (ii) are straightforward. For part (iii), we have to show that

$$\frac{C_P}{C_V} = 1 + \left(\frac{s}{n}\right)^2 \frac{C_V}{Nk} \frac{f_{(n/s)-1}(z)}{f_{n/s}(z)} = \left(1 + \frac{s}{n}\right) \frac{f_{(n/s)+1}(z) f_{(n/s)-1}(z)}{\{f_{n/s}(z)\}^2}, \quad (1)$$

which can be done quite easily; see eqns. (7) - (9) of the solution to Problem 7.14. For part (iv), we observe that, since the quantity S/N is a function of z only, an *isentropic* process implies that $z = \text{const.}$ Accordingly, for such a process,

$$VT^{n/s} = \text{const.} \quad \text{and} \quad P/T^{(n/s)+1} = \text{const.};$$

see eqns. (2) and (3) of the solution to Problem 7.14. Eliminating T among these relations, we obtain the desired equation of an adiabat. For part (v), we proceed as follows.

In this limit $z \rightarrow 0$, eqn. (1) gives

$$C_P / C_V \rightarrow 1 + (s/n). \quad (1a)$$

For $z \gg 1$, on the other hand, we obtain [see formula (E.15)]

$$\begin{aligned} \frac{C_P}{C_V} &= \left\{ 1 + \left(\frac{n}{s} + 1\right) \frac{n}{s} \frac{\pi^2}{6} (\ln z)^{-2} + \dots \right\} \left\{ 1 + \left(\frac{n}{s} - 1\right) \left(\frac{n}{s} - 2\right) \frac{\pi^2}{6} (\ln z)^{-2} + \dots \right\} \times \\ &\quad \left\{ 1 + \frac{n}{s} \left(\frac{n}{s} - 1\right) \frac{\pi^2}{6} (\ln z)^{-2} + \dots \right\}^{-2} \\ &= 1 + \frac{\pi^2}{3} (\ln z)^{-2} + \dots \approx 1 + \frac{\pi^2}{3} (kT/\epsilon_F)^2, \end{aligned}$$

regardless of the values of s and n .

8.12. For a Fermi gas confined to a two-dimensional region of area A ,

$$N = \frac{A}{\lambda^2} f_1(z_F) = \frac{A}{\lambda^2} \ln(1 + z_F), \quad E_F = \frac{AkT}{\lambda^2} f_2(z_F), \quad (1a,b)$$

while the corresponding results for the Bose gas are

$$N = \frac{A}{\lambda^2} g_1(z_B) = -\frac{A}{\lambda^2} \ln(1 - z_B), \quad E_B = \frac{AkT}{\lambda^2} g_2(z_B). \quad (2a,b)$$

Equating (1a) and (2a), we get

$$1 + z_F = \frac{1}{1 - z_B}, \text{ i.e. } z_F = \frac{z_B}{1 - z_B} \text{ or } z_B = \frac{z_F}{1 + z_F}.$$

Next, since $z \partial f_2(z) / \partial z = f_1(z)$,

$$f_2(z_F) = \int_0^{z_F} \frac{1}{z} \ln(1+z) dz = \int_0^{z_F} \left\{ \frac{1}{1+z} + \frac{1}{z(1+z)} \right\} \ln(1+z) dz.$$

The first part of this integral is readily evaluated; in the second part, we substitute $z = z' / (1 - z')$, to get

$$f_2(z_F) = \frac{1}{2} \ln^2(1 + z_F) - \int_0^{z_F/(1+z_F)} \frac{1}{z'} \ln(1 - z') dz' = \frac{1}{2} \ln^2(1 + z_F) + g_2(z_B).$$

Equations (1b) and (2b) then yield the desired result, viz.

$$E_F(N, T) = \frac{N^2 h^2}{4\pi m A} + E_B(N, T), \text{ whence } \{C_V(N, T)\}_F = \{C_V(N, T)\}_B.$$

Letting $T \rightarrow 0$, we recognize that the constant appearing in the above result must be equal to $E_F(N, 0)$. To verify this, we note that, since the Fermi momentum of the gas in two dimensions is given by the equation $N = A \cdot \pi p_F^2 / h^2$, the Fermi energy is given by $\varepsilon_F = p_F^2 / 2m = N h^2 / 2\pi m A$. The ground-state energy of the gas then follows readily:

$$E_F(N, 0) = \int_0^{p_F} \frac{p^2}{2m} \frac{A \cdot 2\pi p dp}{h^2} = \frac{A \cdot \pi p_F^4}{4m h^2} = \frac{N^2 h^2}{4\pi m A} = \frac{1}{2} N \varepsilon_F.$$

8.13. The Fermi energy of the gas is given by the obvious relation

$$N = \int_0^{\varepsilon_F} a(\varepsilon) d\varepsilon. \quad (1)$$

At the same time, the quantities N and U , as functions of μ and T , are given by the standard integrals

$$N = \int_0^{\infty} \frac{a(\varepsilon) d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1} \quad \text{and} \quad U = \int_0^{\infty} \frac{\varepsilon a(\varepsilon) d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1}.$$

At low temperatures we employ formula (E.16), with $x = \beta\varepsilon$ and $\xi = \beta\mu$, to obtain

$$\begin{aligned} N &= \int_0^{\mu} a(\varepsilon) d\varepsilon + \frac{\pi^2}{6} (kT)^2 \left\{ \frac{da(\varepsilon)}{d\varepsilon} \right\}_{\varepsilon=\mu} + \dots \\ &\approx \int_0^{\varepsilon_F} a(\varepsilon) d\varepsilon + (\mu - \varepsilon_F) a(\varepsilon_F) + \frac{\pi^2}{6} (kT)^2 \left\{ \frac{da(\varepsilon)}{d\varepsilon} \right\}_{\varepsilon=\varepsilon_F}, \end{aligned} \quad (2)$$

$$U = \int_0^{\mu} \varepsilon a(\varepsilon) d\varepsilon + \frac{\pi^2}{6} (kT)^2 \left\{ a(\varepsilon) + \varepsilon \frac{da(\varepsilon)}{d\varepsilon} \right\}_{\varepsilon=\mu} + \dots$$

$$\approx \int_0^{\epsilon_F} \epsilon a(\epsilon) d\epsilon + (\mu - \epsilon_F) \epsilon_F a(\epsilon_F) + \frac{\pi^2}{6} (kT)^2 \left\{ a(\epsilon_F) + \epsilon_F \left[\frac{da(\epsilon)}{d\epsilon} \right]_{\epsilon=\epsilon_F} \right\}. \quad (3)$$

Comparing (1) and (2), we obtain for the chemical potential of the gas

$$\mu \approx \epsilon_F - \frac{\pi^2}{6} \frac{(kT)^2}{a(\epsilon_F)} \left\{ \frac{da(\epsilon)}{d\epsilon} \right\}_{\epsilon=\epsilon_F}, \quad (4)$$

which leads to the desired result for μ .

Next, substituting (4) into (3), we obtain the remarkably simple expression

$$U \approx U_0 + (\pi^2 / 6) k^2 T^2 a(\epsilon_F),$$

whence

$$C_V \approx (\pi^2 / 3) k^2 T a(\epsilon_F). \quad (5)$$

It follows that

$$S = \int_0^T \frac{C_V dT}{T} \approx (\pi^2 / 3) k^2 T a(\epsilon_F). \quad (6)$$

For a gas with energy spectrum $\epsilon \propto p^s$, confined to a space of n dimensions,

$$a(\epsilon) d\epsilon \sim p^{n-1} dp \sim \epsilon^{(n/s)-1} d\epsilon.$$

By eqn. (1), the Fermi energy of the gas is given by

$$N = \int_0^{\epsilon_F} A \epsilon^{(n/s)-1} d\epsilon = \frac{sA}{n} \epsilon_F^{n/s} = \frac{s \epsilon_F}{n} a(\epsilon_F).$$

Substituting this result into (5), we get

$$\frac{C_V}{Nk} \approx \frac{n}{s} \cdot \frac{\pi^2}{3} \left(\frac{kT}{\epsilon_F} \right); \quad (7)$$

cf. eqn. (8.1.39), which pertains to the case $n = 3, s = 2$. See also Problem 8.11.

8.19. Utilizing the result obtained in Problem 8.13, we have for a Fermi gas at low temperatures

$$\frac{C_V}{Nk} = \frac{\pi^2}{3} \frac{a(\varepsilon_F)}{N} kT \quad (1)$$

Now, the density of states for the relativistic gas is given by, see eqn. (8.4.7),

$$a(\varepsilon) = \frac{8\pi V}{h^3} p^2 \frac{dp}{d\varepsilon} = \frac{8\pi mV}{h^3} p \left\{ 1 + \left(\frac{p}{mc} \right)^2 \right\}^{1/2},$$

where $p = p(\varepsilon)$. Substituting this result into (1) and making use of eqn. (8.4.4), we get

$$\frac{C_V}{Nk} = \frac{\pi^2 m}{p_F^2} \left\{ 1 + \left(\frac{p_F}{mc} \right)^2 \right\}^{1/2} kT,$$

which leads to the desired result.

In the non-relativistic case ($p_F \ll mc$ and $\varepsilon_F = p_F^2 / 2m$), we obtain the familiar expression (8.1.39); in the extreme relativistic case ($p_F \gg mc$ and $\varepsilon = pc$), we obtain

$$\frac{C_V}{Nk} = \pi^2 \left(\frac{kT}{\varepsilon_F} \right),$$

consistent with expression (7) of the solution to Problem 8.13.