Spring 2003 Physics 603 O. W. Greenberg

Solutions to homework due 4/22/03

Assignment 8

7.11. Under the conditions of this problem, the summation in eqn. (7.1.2) has to be carried out over the states of the *internal* spectrum as well as over the translational states. Expression (7.1.16) is then replaced by

$$N_e = (N_e)_0 + (N_e)_1 = \frac{V}{\lambda^3} g_{3/2}\left\{\exp\left(\frac{\mu}{kT}\right)\right\} + \frac{V}{\lambda^3} g_{3/2}\left\{\exp\left(\frac{\mu-\varepsilon_1}{kT}\right)\right\}.$$

The critical temperature T_c is then determined by the condition

$$\frac{V}{\lambda_c^3} g_{3/2}(1) + \frac{V}{\lambda_c^3} g_{3/2}(x) = N, \text{ where } x = e^{-\varepsilon_1/kT_c}. \qquad (1)$$

For $x \ll 1$, $g_{3/2}(x) \simeq x$ and eqn. (1) gives

$$\lambda_c^3 \simeq (V/N)[\zeta(3/2) + x].$$

Comparing this with the standard result $\left(\lambda_c^0\right)^3 = (V/N)\zeta(3/2)$, we get

$$\frac{T_c}{T_c^0} \equiv \left(\frac{\lambda_c^0}{\lambda_c}\right)^2 \simeq \left[1 + \frac{x}{\zeta(3/2)}\right]^{-2/3} \simeq 1 - \frac{2/3}{\zeta(3/2)} x \simeq 1 - \frac{2/3}{\zeta(3/2)} e^{-\varepsilon_1/kT_c^0}.$$

For $x \lesssim 1$, on the other hand, $g_{3/2}(x) \simeq \zeta(3/2) - 2\pi^{1/2}(-\ell n\, x)^{1/2}$; eqn. (1) now gives

$$\lambda_c^3 \simeq (2V/N)\left[\zeta(3/2) - \pi^{1/2}(\varepsilon_1/kT_c)^{1/2}\right], \text{ whence}$$

$$\frac{T_c}{T_c^0} \simeq \left\{2\left[1 - \frac{\pi^{1/2}}{\zeta(3/2)}\left(\frac{\varepsilon_1}{kT_c}\right)^{1/2}\right]\right\}^{-2/3} \simeq 2^{-2/3}\left[1 + \frac{2}{3}\frac{\pi^{1/2}}{\zeta(3/2)} 2^{1/3}\left(\frac{\varepsilon_1}{kT_c^0}\right)^{1/2}\right].$$

7.13. It is straightforward to see that for a Bose gas in two dimensions

$$N_e = \int_0^\infty \frac{1}{z^{-1}e^{\beta\varepsilon}-1}\frac{A\cdot 2\pi p\,dp}{h^2} = \frac{A\cdot 2\pi mkT}{h^2}\int_0^\infty \frac{dx}{z^{-1}e^x-1} = \frac{A}{\lambda^2}g_1(z)\,,$$

while

$$N_0 = \frac{z}{1-z}\,.$$

Since Bose-Einstein condensation requires that $z \to 1$, the critical temperature T_c, by the usual argument, is given by

$$\left(\frac{N}{A}\right)\lambda_c^2 = g_1(1) = \infty \qquad [\text{ for } g_1(z) = -\ell n\,(1-z)]\,.$$

It follows that $T_c = 0$.

More accurately, the phenomenon of condensation requires that both N_e and N_0 be of order N. This means that, while $z \simeq 1$, $(1-z)$ be of order N^{-1} and hence λ^2 be of order $(A\,\ell n N / N)$. Since the ratio $(A/N) \sim \ell^2$, the condition for condensation takes the form $(\lambda^2/\ell^2) = O(\ell n\,N)$. It follows that

$$T \equiv \frac{h^2}{2\pi mk\lambda^2} \sim \frac{h^2}{mk\,\ell^2}\frac{1}{\ell n\,N}\,.$$

7.14. With energy spectrum $\varepsilon = Ap^s$, the density of states in the system is given by, see formula (C.7b),

$$a(\varepsilon)d\varepsilon = \frac{V}{h^n}\frac{2\pi^{n/2}}{\Gamma(n/2)}p^{n-1}dp = \frac{V}{h^n}\frac{2\pi^{n/2}}{sA^{n/s}\Gamma(n/2)}\varepsilon^{(n/s)-1}d\varepsilon \,. \tag{1}$$

This leads to the expression

$$\begin{aligned} N - N_0 &= \frac{V}{h^n}\frac{2\pi^{n/2}}{sA^{n/s}\Gamma(n/2)}\int_0^\infty \frac{\varepsilon^{(n/s)-1}}{z^{-1}e^{\beta\varepsilon}-1}d\varepsilon \\ &= \frac{V}{h^n}\frac{2\pi^{n/2}\Gamma(n/s)}{sA^{n/s}\Gamma(n/2)}(kT)^{n/s}g_{n/s}(z) \,, \end{aligned} \tag{2}$$

while $N_0 = z/(1-z)$. Similarly,

$$P = \frac{1}{h^n}\frac{2\pi^{n/2}\Gamma(n/s)}{sA^{n/s}\Gamma(n/2)}(kT)^{(n/s)+1}g_{(n/s)+1}(z) \,. \tag{3}$$

Next, following the derivation of eqn. (7.1.11), we get

$$U = kT^2\left\{\frac{\partial}{\partial T}\left(\frac{PV}{kT}\right)\right\}_{z,V} = \frac{n}{s}PV \,, \tag{4}$$

so that $P = sU/nV$.

The onset of Bose-Einstein condensation requires that $z \to 1$ at a finite temperature T_c. A glance at eqn. (2) tells us that this will happen only if $n > s$ and that the critical temperature T_c will then be determined by the equation

$$N = \frac{V}{h^n}\frac{2\pi^{n/2}\Gamma(n/s)}{sA^{n/s}\Gamma(n/2)}(kT_c)\zeta\left(\frac{n}{s}\right) . \tag{5}$$

For $T < T_c$, N_e will be equal to $N(T/T_c)^{n/s}$ while N_0 will be given by the balance $(N - N_e)$.

To study the specific heats we first observe, from eqns. (2) - (4), that for $T > T_c$ (when $N_0 \ll N$)

$$U = \frac{n}{s}NkT \cdot g_{(n/s)+1}(z)/g_{n/s}(z) \,. \tag{6}$$

Next, using eqns. (2) and (3), and the recurrence relation (D.10), we get

$$\frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_V = -\frac{n}{s}\frac{1}{T}\frac{g_{n/s}(z)}{g_{(n/s)-1}(z)} \quad \text{and} \quad \frac{1}{z}\left(\frac{\partial z}{\partial T}\right)_P = -\left(\frac{n}{s}+1\right)\frac{1}{T}\frac{g_{(n/s)+1}(z)}{g_{n/s}(z)} \,. \tag{7}$$

It is now straightforward to show that

$$\frac{C_V}{Nk} = \frac{n}{s}\left(\frac{n}{s}+1\right)\frac{g_{(n/s)+1}(z)}{g_{n/s}(z)} - \left(\frac{n}{s}\right)^2\frac{g_{n/s}(z)}{g_{(n/s)-1}(z)} \tag{8}$$

and

$$\frac{C_P}{Nk} = \left(\frac{n}{s}+1\right)^2\frac{\{g_{(n/s)+1}(z)\}^2 g_{(n/s-1)}(z)}{\{g_{n/s}(z)\}^3} - \frac{n}{s}\left(\frac{n}{s}+1\right)\frac{g_{(n/s)+1}(z)}{g_{n/s}(z)} \,. \tag{9}$$

The limiting cases suggested in the problem follow quite easily.

7.19. According to Sec. 7.3,

$$C_V(T) = \int_\omega \frac{\partial}{\partial T}\left\{\frac{\hbar\omega}{e^{\hbar\omega/kT}-1}\right\} g(\omega)d\omega, \quad \text{while} \quad C_V(\infty) = \int_\omega k\, g(\omega)d\omega \, .$$

It follows that

$$\int_0^\infty \{C_V(\infty) - C_V(T)\}\, dT = \int_\omega \left[kT - \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}\right]_0^\infty g(\omega)d\omega \, .$$

It is easy to show that

$$\lim_{T\to\infty} \frac{\hbar\omega}{e^{\hbar\omega/kT}-1} \approx kT - \frac{1}{2}\hbar\omega \, ;$$

see page 69 as well as Fig. 3.4 of the text. The intgral on the right-hand side then becomes

$$\int_\omega \frac{1}{2}\hbar\omega \cdot g(\omega)d\omega \, ,$$

which is indeed equal to the zero-point energy of the solid.

The physical interpretation of this result lies in noting that the actual amount of heat required to raise the temperature of a solid is less than the value predicted classically because the solid already possesses a finite amount of energy even at $T = 0K$.