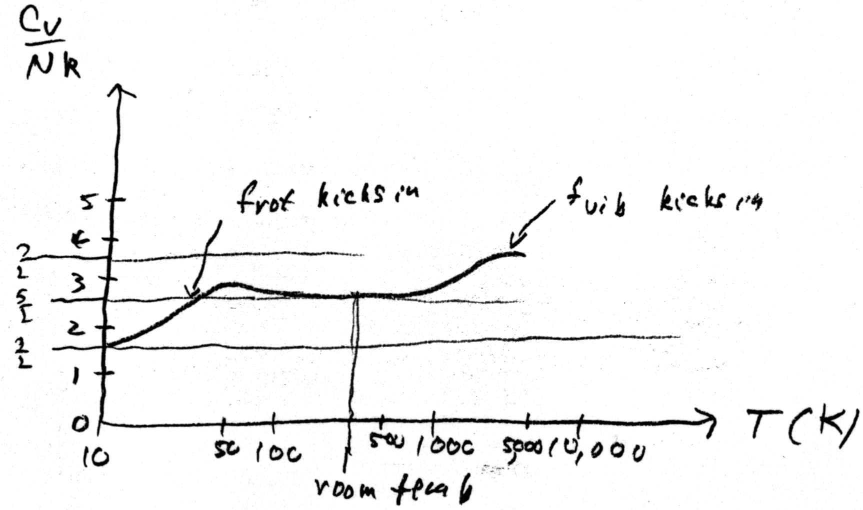
Physics 603 Spring 2003 O. W. Greenberg

Solutions to Exam Two

1. (a)



$$\frac{C_V}{Nk} = \frac{1}{2} f, \quad f = \text{no. of degrees of freedom excited}$$

$$f_{transl.} = 3, \quad f_{rot.} = 2, \quad f_{vib} = 2$$

(b) $\psi_{tot} = \psi_{cm}\left(\frac{r_I + r_{II}}{2}\right) \psi_{el}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_Z; \sigma_1, \sigma_2, \ldots, \sigma_Z; \vec{r}_I, \vec{r}_{II}) Y_{rot}(\vec{r}_I - \vec{r}_{II}) \psi_{nuc\text{-}spin}(\sigma_I, \sigma_{II})$

$\psi_{el} = \hat{\psi}_{el}(\vec{r}_i - \vec{r}_I, \vec{r}_i - \vec{r}_{II}) \psi_{el\text{-}spin}(\sigma_i)$ $\quad \vec{r}_i, \sigma_i$ electrons

$\psi_{rot} \propto Y^J_M(\theta, \phi)$ $\quad \vec{r}_{I,II}, \sigma_{I,II}$ nuclei

$$\psi_{nuc\text{-}spin} = \begin{cases} \text{sym, weight} & (s_A+1)(2s_A+1) \\ \text{antisym,} & (s_A)(2s_A+1) \end{cases}$$

(c) $P_{I,II} \hat{\psi}_{el} = +\hat{\psi}_{el}$ because Pauli principle is taken account of by $\psi_{el\text{-}spin}$ and $\hat{\psi}_{el}$ is nodeless and therefore sym. under $P_{I,II}$.

$P_{I,II} \psi_{cm} = +\psi_{cm}$ obviously

so $P_{I,II} \psi_{tot}$ comes from $P_{I,II} \psi_{rot} \cdot \psi_{nuc\text{-}spin}$

(d) If N^{14} has 7 e^- and 14 p^+ then it is a fermion,

so $P_{I,II}\, \psi_{rot}\, \psi_{nuc-spin} = -\psi_{rot}\, \psi_{nuc-spin}$ $\qquad s_A = 1$

$$\jmath_{nuclei} = \underbrace{S_A(2S_A+1)}_{3} \sum_{J\ even} (2J+1)\, e^{-J(J+1)\frac{\theta}{T}} +$$

$$+ \underbrace{(S_A+1)(2S_A+1)}_{6} \sum_{J\ odd} (2J+1)\, e^{-J(J+1)\frac{\theta}{T}}$$

If N^{14} has 7 n^0 and 7 p^+ then it is a boson,

so $P_{I,II}\, \psi_{rot}\, \psi_{nuc-spin} = +\psi_{rot}\, \psi_{nuc-spin}$

$$\jmath_{nuclei} = \underbrace{(S_A+1)(2S_A+1)}_{6} \sum_{J\ even} (2J+1)\, e^{-J(J+1)\frac{\theta}{T}}$$

$$+ \underbrace{S_A(2S_A+1)}_{3} \sum_{J\ odd} (2J+1)\, e^{-J(J+1)\frac{\theta}{T}}$$

At low T, only the lowest state $J=0$ is populated, so

for 7 e^-, 14 p^+, have the antisym. spin state: $S_{nuclei} = 1$

for 7 n^0, 7 p^+, " " sym. " " : $S_{nuclei} = 0$ or 2

2. (a) Use $\mathcal{Q} = \sum_i z^{\sum n_i} e^{-\beta \sum n_i \epsilon_i}$

$$N = z\frac{\partial}{\partial z} \ln \mathcal{Q} = \frac{\sum_i \sum n_i\, z^{\sum n_i} e^{-\beta \sum n_i \epsilon_i}}{\sum_i z^{\sum n_i} e^{-\beta \sum n_i \epsilon_i}}$$

$$z\frac{\partial}{\partial z} \ln \mathcal{Q} = z\frac{\partial}{\partial z} \sum_i (-1) \ln (1 - z e^{-\beta \epsilon_i})$$

$$= \sum_i \frac{(-1)(-1)\, z e^{-\beta \epsilon_i}}{1 - z e^{-\beta \epsilon_i}} = \sum_i \frac{1}{\frac{1}{z} e^{\beta \epsilon_i} - 1}$$

$$\frac{1}{h^3}\, d^3p\, d^3q \to \frac{1}{h^3}\, 4\pi p^2 dp\, V \qquad \frac{p^2}{2m} = \epsilon, \quad \frac{p\,dp}{m} = d\epsilon$$

$$= \frac{1}{h^3}\, 4\pi \sqrt{2m\epsilon}\; m\, d\epsilon\, V$$

$$N = \frac{4\pi\sqrt{2}\, V\, m^{3/2}}{h^3} \int_0^\infty \frac{\sqrt{\epsilon}\, d\epsilon}{\frac{1}{z}e^{\beta\epsilon}-1} + \underbrace{\frac{1}{\frac{1}{z}-1}}_{\frac{z}{1-z}}$$

Let $x = \beta\epsilon$

$$N = \frac{4\pi\sqrt{2}}{h^3}\, m^{3/2} (kT)^{3/2}\, V \int_0^\infty \frac{\sqrt{x}\, dx}{\frac{1}{z}e^x - 1} + \frac{z}{1-z}$$

$\int_0^\infty \frac{\sqrt{x}\, dx}{\frac{1}{z}e^x - 1}$ increases monotonically for $z \to 1$, so the maximum is $\int_0^\infty \frac{\sqrt{x}\,dx}{e^x - 1} \equiv I$

when $\frac{N}{V} = n > \frac{2\pi\,(2mkT)^{3/2}}{h^3} I$, need condensate, so

$$\frac{2\pi\,(2mkT_c)^{3/2} I}{h^3} = n, \text{ or}$$

(b) $$T_c = \frac{h^2\, n^{2/3}}{(2\pi)^{2/3}\, 2mk\, I^{2/3}}$$

(c) If $N \sim \int \frac{\sqrt{\epsilon}\, d\epsilon}{\frac{1}{z}e^{\beta\epsilon}-1}$, $E \sim \int \frac{\epsilon\sqrt{\epsilon}\, d\epsilon}{\frac{1}{z}e^{\beta\epsilon}-1}$. Then $x = \beta\epsilon$ will give $E \sim T^{5/2}$ for the non-condensate particles (only these contribute to E), so $C_V = \frac{\partial E}{\partial T} \sim T^{3/2}$ for $T < T_c$.