It is easy to check that

$$\sigma_j \sigma_k = -\sigma_k \sigma_j \quad \text{in general for } j,k = 1 \text{ to } 3, \; j \neq k$$

$$\therefore \quad \sigma_j \sigma_k = \delta_{jk} \sigma_0 + i \sum_{\ell} \epsilon_{jk\ell} \sigma_\ell$$

$$\therefore \quad (\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = \sum_{jk} a_j b_k \sigma_j \sigma_k$$

$$= \sum_{jk} a_j b_k \delta_{jk} \sigma_0 + i \sum_{jk\ell} a_j b_k \epsilon_{jk\ell} \sigma_\ell$$

$$= (\vec{a} \cdot \vec{b}) \sigma_0 + i \sum_{jk\ell} \epsilon_{jk\ell} a_j b_k \hat{e}_\ell \cdot \vec{\sigma}$$

$$= (\vec{a} \cdot \vec{b}) \sigma_0 + i (\vec{a} \times \vec{b}) \cdot \vec{\sigma}$$

d) $$[J_1, J_2] = \left(-\frac{i}{2}\right)\left(-\frac{i}{2}\right) [\sigma_1, \sigma_2]$$

$$= \left(-\frac{1}{4}\right)(\sigma_1 \sigma_2 - \sigma_2 \sigma_1) = \left(-\frac{1}{4}\right)(\sigma_1 \sigma_2 + \sigma_1 \sigma_2)$$

$$= \left(-\frac{1}{4}\right)(2)(i \sigma_3) = -\frac{i}{2} \sigma_3 = J_3, \text{ etc.}$$