Let $f(\lambda)$ be the function

$$f(\lambda) = \det[\exp(\lambda B)].$$

Verify the expansion

$$\begin{aligned} f(\lambda + d\lambda) &= \det\{\exp[(\lambda + d\lambda)B]\} \\ &= \det[\exp(\lambda B)\exp(d\lambda B)] \\ &= \underbrace{\det[\exp(\lambda B)]}_{= f(\lambda)}\,\underbrace{\det[\exp(d\lambda B)]}_{= I + d\lambda B + O[(d\lambda)^2]} \\ &= f(\lambda)\det\{1 + d\lambda B + O[(d\lambda)^2]\} \\ &= f(\lambda)\{1 + d\lambda \mathrm{tr}(B) + O[(d\lambda)^2]\}. \quad ☆ \end{aligned}$$

Show that $f(\lambda)$ obeys the differential equation

$$df/d\lambda = f(\lambda)\mathrm{tr}(B)$$

with the initial condition

$$f(0) = 1.$$

From ☆ we have

$$f(\lambda + d\lambda) - f(\lambda) = f(\lambda)\, d\lambda\, \mathrm{tr}(B) + O[(d\lambda)^2] \Rightarrow$$

$$\frac{f(\lambda + d\lambda) - f(\lambda)}{d\lambda} = f(\lambda)\,\mathrm{tr}(B) + O[(d\lambda)^2] \Rightarrow$$

$$\frac{df}{d\lambda} = f(\lambda)\,\mathrm{tr}(B). \quad ☆☆$$

Also, $f(0) = \det[\exp(0)] = \det(I) = 1.$