63.

7.4: Verify the relations given by Eqs. (7.3) through (7.6) using the definitions (7.1) and (7.2).

3.7 Lie Algebraic Properties

Let A be any matrix. The exponential of a matrix, written variously as e^A or $\exp(A)$, is defined by the exponential series

$$e^A = \exp(A) = \sum_{n=0}^{\infty} A^n/n!. \tag{3.7.1}$$

Similarly, the logarithm of a matrix A (sufficiently near the identity) is defined by the series

$$\log(A) = \log[I - (I - A)] = -\sum_{n=1}^{\infty}(I - A)^n/n. \tag{3.7.2}$$

As might be expected, the exponential and logarithmic functions are related. Specifically, if one has

$$B = \log(A), \tag{3.7.3}$$

then it follows that

$$A = \exp(B). \tag{3.7.4}$$

Put another way, one has the relations

$$A = \exp[\log(A)] \text{ for } A \text{ sufficiently near the identity matrix,} \tag{3.7.5}$$

$$B = \log[\exp(B)] \text{ for } B \text{ sufficiently near the zero matrix.} \tag{3.7.6}$$

To verify (7.5), let x be an ordinary number. Then we have

$$x \stackrel{?}{=} \exp[\log(x)] \Rightarrow$$

$$x \stackrel{?}{=} \exp\left[(-1)\sum_{n=1}^{\infty}(1-x)^n/n\right] \Rightarrow$$

$$x \stackrel{?}{=} 1 + [\quad] + [\quad]^2/2! + [\quad]^3/3! + \cdots \Rightarrow$$

$$x \stackrel{?}{=} 1 + (-1)\left\{\frac{(1-x)}{1} + \frac{(1-x)^2}{2} + \cdots\right\} + \frac{(-1)^2}{2!}\left\{(1-x) + \frac{(1-x)^2}{2} + \cdots\right\}^2$$

$$\Rightarrow x = \underbrace{1 - (1-x)}_{\text{cancel}} \underbrace{- \frac{(1-x)^2}{2} + \cdots + \frac{(1-x)^2}{2}}_{\text{cancel}} + \cdots \Rightarrow$$

$x = x +$ all remaining terms cancel.