for small excursions from a circular orbit in the $z = 0$ plane. Consequently,

$$L = \left(\tfrac{1}{2}\right) m\left[\dot{r}^2 + (r\dot{\phi})^2\right] + \frac{mMG}{r} + \frac{1}{2} m\dot{z}^2 - \frac{mMGz^2}{2R^3} + O(\epsilon^3),$$

and the z motion uncouples in lowest order. That is, in lowest approximation, the z motion is governed by the Lagrangian

$L_z = \frac{1}{2} m\dot{z}^2 - \frac{1}{2}\left(\frac{mMG}{R^3}\right) z^2$. This Lagrangian produces the equation of motion

$$m\ddot{z} + \frac{mMG}{R^3} z = 0, \quad \text{or} \quad \boxed{\ddot{z} + \omega^2 z = 0}$$

with $\omega^2 = \frac{MG}{R^3}$ as before. Thus, z is also periodic with the same period as ρ, etc. Consequently, the lens cap again comes back to the cosmonaut at $t = \tau = 2\pi/\omega$.