From (4.56) we have

$$\det[L(t+dt)] - \det[L(t)] = dt\, \mathrm{tr}[A(t)]\, \det[L(t)] + O[(dt)^2] \Rightarrow$$

$$\frac{\det[L(t+dt)] - \det[L(t)]}{dt} = \mathrm{tr}[A(t)]\, \det[L(t)] + O[(dt)] \Rightarrow$$

$$\frac{d}{dt} \det[L(t)] = \{\mathrm{tr}[A(t)]\}\{\det[L(t)]\}.$$

This proves (4.57). Next we may write

$$\frac{\frac{d}{dt} \det[L(t)]}{\det[L(t)]} = \mathrm{tr}[A(t)] \xRightarrow{\text{integrating}}$$

$$\log\{\det[L(t')]\}\Big|_{t^i}^{t} = \int_{t^i}^{t} dt'\, \mathrm{tr}[A(t')].$$

Also, $L(t^i) = I \Rightarrow \log\{\det[L(t^i)]\} = \log 1 = 0$

$$\Rightarrow \det[L(t)] = \exp\{\int_{t^i}^{t} dt'\, \mathrm{tr}[A(t')]\},$$

which is (4.58). In particular,