From (4.46) we have   60 cont.   3/5

$$\vec{y}^f = \vec{y}(t^f) = \vec{y}^d(t^f) + \epsilon\, \vec{\eta}(t^f)$$

$$= \vec{y}^d(t^f) + \epsilon\, L(t^f)\, \vec{\eta}^i + O(\epsilon^2)$$

$$= \vec{y}^d(t^f) + L(t^f)\, \epsilon \vec{\eta}^i + O(\epsilon^2)$$

But, $\epsilon \vec{\eta}^i = \vec{y}(t^i) - \vec{y}^d(t^i) = \vec{y}^i - \vec{y}^d(t^i) = \delta \vec{y}^i$

and $\vec{y}^f - \vec{y}^d(t^f) = \delta \vec{y}^f \Rightarrow$

$$\delta \vec{y}^f = L(t^f)\, \delta \vec{y}^i + O(\epsilon^2) \Rightarrow$$

$$M_{jk} = \partial y_j^f / \partial y_k^i = L_{jk}(t^f).$$

Use (4.51) to write the Taylor expansion

$$\begin{aligned} L(t+dt) &= L(t) + \dot{L}(t)dt + O[(dt)^2] \\ &= L(t) + dtA(t)L(t) + O[(dt)^2] \\ &= [I + dtA(t)]L(t) + O[(dt)^2]. \end{aligned} \tag{1.4.55}$$

Take determinants of both sides of (4.55) to get the result

$$\begin{aligned} \det[L(t+dt)] &= \{\det[I + dtA(t)]\} \det[L(t)] + O[(dt)^2] \\ &= \{1 + dt\ \mathrm{tr}[A(t)]\} \det[L(t)] + O[(dt)^2]. \end{aligned} \tag{1.4.56}$$

Here use has been made of (3.7.95). Show that (4.56) produces the differential equation

$$(d/dt)\det[L(t)] = \{\mathrm{tr}[A(t)]\}\{\det[L(t)]\} \tag{1.4.57}$$

and, in view of (4.52), that this equation has the explicit solution

$$\det[L(t)] = \exp\{\int_{t^i}^{t} dt'\ \mathrm{tr}[A(t')]\}. \tag{1.4.58}$$

In particular, there is the result

$$\det(M) = \exp\{\int_{t^i}^{t^f} dt\ \mathrm{tr}[A(t)]\}. \tag{1.4.59}$$