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4.6: Let $\mathcal{M}$ be a map of an m-dimensional space into itself as in (4.6). Associated with this map is an $m \times m$ matrix $M(y^i)$, called the *Jacobian* matrix of $\mathcal{M}$, defined by the rule

$$M_{jk}(y^i) = \partial y_j^f / \partial y_k^i. \tag{1.4.43}$$

This matrix describes how small changes in the initial conditions y^i produce small changes in the final conditions y^f. Note that generally the Jacobian matrix depends on the initial conditions, and therefore we write $M(y^i)$. In the case that $\mathcal{M}$ is a transfer map arising from a differential equation as in (4.1), the associated Jacobian matrix can be found by integrating the *variational* equations derived from (4.1). Let y^i be a set of initial conditions and let $y^d(t)$ be the trajectory (sometimes called the *design* trajectory) that has these initial conditions

$$y^d(t^i) = y^i. \tag{1.4.44}$$

Because it is a trajectory, it satisfies the differential equation

$$\dot{y}^d = f(y^d; t). \tag{1.4.45}$$

Next condsider nearby trajectories of the form

$$y(t) = y^d(t) + \epsilon\eta(t) \tag{1.4.46}$$

where ϵ is small. Insertion of (4.45) into (4.1) gives the equation

$$\dot{y}^d + \epsilon\dot{\eta} = f(y^d + \epsilon\eta; t). \tag{1.4.47}$$

Now take components of both sides of (4.47) and expand in powers of ϵ to find the relation

$$\dot{y}_j^d + \epsilon\dot{\eta}_j = f_j(y^d; t) + \sum_k [(\partial f_j / \partial y_k)\big|_{y=y^d}]\epsilon\eta_k + O(\epsilon^2)\,. \tag{1.4.48}$$

Define the $m \times m$ matrix $A(t)$ by the rule

$$A_{jk}(t) = (\partial f_j / \partial y_k)\big|_{y=y^d}\,. \tag{1.4.49}$$

Use (4.45), (4.48), and (4.49) to show that η, in the limit $\epsilon \to 0$, satisfies the set of equations

$$\dot{\eta} = A(t)\eta. \tag{1.4.50}$$

Subtracting (4.45) from (4.48) gives

$$\epsilon\,\dot{\eta}_j = \epsilon \sum_k \partial f_j/\partial y_k \, \eta_k + O(\epsilon^2) \quad \star$$

Using (4.49) in $\star$ gives