$$\mathcal{H} = -f^{-\frac{1}{2}}\Big\{ r^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + \dot r^2\left(1-\tfrac{a}{r}\right)^{-1}$$
$$+ c^2\left(1-\tfrac{a}{r}\right) - \dot r^2\left(1-\tfrac{a}{r}\right)^{-1} - r^2(\dot\theta^2+\sin^2\theta\,\dot\phi^2)\Big\}$$

$$\therefore \mathcal{H} = -f^{-\frac{1}{2}}c^2\left(1-\tfrac{a}{r}\right) = \text{cnst} = -W/c$$

Let us write out these constants in more detail using $\theta = \pi/2$ & $\dot\theta = 0$. We get

i) $$L/c = r^2\dot\phi\, f^{-\frac{1}{2}} = r^2\dot\phi\left\{c^2\left(1-\tfrac{a}{r}\right) - \dot r^2\left(1-\tfrac{a}{r}\right)^{-1} - r^2\dot\phi^2\right\}^{-\frac{1}{2}}$$

ii) $$W/c = f^{-\frac{1}{2}}c^2\left(1-\tfrac{a}{r}\right)$$

i) ÷ ii) $\Rightarrow$ $$\frac{L}{c}\frac{c}{W} = r^2\dot\phi\, f^{-\frac{1}{2}}\, f^{\frac{1}{2}}\, c^{-2}\left(1-\tfrac{a}{r}\right)^{-1}$$

or $$r^2\dot\phi\, c^{-2}\left(1-\tfrac{a}{r}\right)^{-1} = \frac{L}{W}$$

or $$\boxed{\dot\phi = \frac{c^2L}{r^2W}\left(1-\tfrac{a}{r}\right)} \qquad \text{I}$$

Square ii) & insert I $\Rightarrow$

$$\left\{c^2\left(1-\tfrac{a}{r}\right) - \dot r^2\left(1-\tfrac{a}{r}\right)^{-1} - \frac{c^4L^2}{r^2W^2}\left(1-\tfrac{a}{r}\right)^2\right\} = \frac{c^2}{W^2}c^4\left(1-\tfrac{a}{r}\right)^2$$

Solve for $\dot r^2$ $\Longrightarrow$