Try $\theta(t) = \pi/2 \Rightarrow \ddot{\Theta} = \ddot{\theta} = 0$ + $\cos\theta = 0$.

Then a) is satisfied with $r^2\dot{\phi} \neq \infty$
and hence $r^2 \sin\theta \cos\theta \dot{\phi}^2 = 0$

Thus b) is also satisfied.

Finally, there is no inconsistency with c)

Thus motion in $\theta = \pi/2$ plane is a solution. Since original Lagrangian is rotationally invariant + eqns of motion are second order, motion started in any plane will remain in a plane. But all initial conditions are equivalent to starting in some plane. Therefore motion always occurs in a plane.

So far, we have one constant of motion. By inspection, $\partial\mathcal{L}/\partial t = 0$. Thus, using Goldstein, page 53, we get a 2nd constant of motion.

$$\mathcal{H} = \sum \dot{q}_i p_i - \mathcal{L} = \text{cnst} \quad \text{where } p_i = \partial\mathcal{L}/\partial\dot{q}_i$$

$\dot{\phi}\, \partial\mathcal{L}/\partial\dot{\phi} + \dot{\theta}\, \partial\mathcal{L}/\partial\dot{\theta} + \dot{r}\, \partial\mathcal{L}/\dot{r} =$

$-f^{-1/2}\left[ r^2(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + \dot{r}^2(1-\frac{a}{r})^{-1}\right]$. Therefore,

$$\mathcal{H} = -f^{-\frac{1}{2}}\left[ r^2(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + \dot{r}^2(1-\frac{a}{r})^{-1}\right] - f^{\frac{1}{2}}$$

or