$$\mathcal{L} = \left\{ c^2\left(1-\frac{a}{r}\right) - \dot r^2 \left(1-\frac{a}{r}\right)^{-1} - r^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) \right\}^{\frac12} = \sqrt{f} \approx c$$

$$\partial\mathcal{L}/\partial\dot\phi = \tfrac12 f^{-\frac12}(-r^2\, 2\dot\phi \sin^2\theta) = -f^{-\frac12} r^2 \sin^2\theta\,\dot\phi$$

$$\partial\mathcal{L}/\partial\phi = 0$$

$$\partial\mathcal{L}/\partial\dot\theta = \tfrac12 f^{-\frac12}(-r^2\, 2\dot\theta) = -f^{-\frac12} r^2\dot\theta$$

$$\partial\mathcal{L}/\partial\theta = \tfrac12 f^{-\frac12}(-r^2\, 2\sin\theta\cos\theta\,\dot\phi) = -f^{-\frac12} r^2 \sin\theta\cos\theta\,\dot\phi^2$$

$$\partial\mathcal{L}/\partial\dot r = \tfrac12 f^{-\frac12}\left[-\left(1-\frac{a}{r}\right)^{-1} 2\dot r\right] = -f^{-\frac12}\left(1-\frac{a}{r}\right)^{-1}\dot r$$

$$\partial\mathcal{L}/\partial r = \tfrac12 f^{-\frac12}\left[c^2\frac{\partial}{\partial r}\left(1-\frac{a}{r}\right) - \dot r^2 \frac{\partial}{\partial r}\left(1-\frac{a}{r}\right)^{-1} - 2r(\dot\theta^2+\sin^2\theta\,\dot\phi^2)\right]$$

Eqns of motion:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \Rightarrow \frac{d}{dt}\left(-f^{-\frac12} r^2\sin^2\theta\,\dot\phi\right) = 0$$

or $$\boxed{f^{-\frac12} r^2\sin^2\theta\,\dot\phi = L/c = \text{const}} \quad a)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \Rightarrow \frac{d}{dt}\left(f^{-\frac12} r^2\dot\theta\right) - f^{-\frac12} r^2\sin\theta\cos\theta\,\dot\phi = 0$$

or $$\boxed{f^{-\frac12} r^2\ddot\theta + \dot\theta\,\frac{d}{dt}\left(f^{-\frac12} r^2\right) - f^{-\frac12} r^2\sin\theta\cos\theta\,\dot\phi^2 = 0} \quad b)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot r} - \frac{\partial\mathcal{L}}{\partial r} = 0 \Rightarrow \frac{d}{dt}\left(f^{-\frac12}\left(1-\frac{a}{r}\right)\dot r\right) + \frac{\partial\mathcal{L}}{\partial r} = 0 \quad \text{or}$$

$$\boxed{f^{-\frac12}\left(1-\frac{a}{r}\right)\ddot r + \dot r\,\frac{d}{dt}\left(f^{-\frac12}\left(1-\frac{a}{r}\right)\right) + \tfrac12 f^{-\frac12}[\ \ ] = 0} \quad c)$$