$$\dot{\theta} = \frac{L}{mr^2} \Rightarrow \dot{\theta} = \frac{L}{m} \frac{1}{\left\{ r_0^2 + 2\left[r_0^2 - \frac{\alpha}{E}\right]^{1/2} \sqrt{\frac{2E}{m}}(t-t_0) + \frac{2E}{m}(t-t_0)^2 \right\}}$$

$$\theta - \theta_0 = \frac{L}{m} \int_{t_0}^{t} \frac{dt'}{\left\{ r_0^2 + 2\left[r_0^2 - \alpha/E\right]^{1/2} \sqrt{\frac{2E}{m}}(t'-t_0) + \frac{2E}{m}(t'-t_0)^2 \right\}} \Rightarrow t' = t_0 + \tau$$

$$\theta - \theta_0 = \frac{L}{m} \int_0^{t-t_0} \frac{d\tau}{\left\{ r_0^2 + 2\left[r_0^2 - \alpha/E\right] \sqrt{\frac{2E}{m}}\, \tau + \frac{2E}{m} \tau^2 \right\}} \Rightarrow$$

$$\theta - \theta_0 = \frac{L}{m} \int_0^{t-t_0} \frac{d\tau}{\left\{ a + b\tau + c\tau^2 \right\}}$$ with

$$a = r_0^2, \quad b = 2\left[r_0^2 - \alpha/E\right]^{1/2} \sqrt{\frac{2E}{m}}, \quad c = \frac{2E}{m}$$

Since $q = 4ac - b^2 < 0$, this integral has the value

$$\boxed{\theta - \theta_0 = \frac{L}{m} \left\{ \frac{-2}{\sqrt{-q}} \tanh^{-1}\left[\frac{(2c\tau + b)}{\sqrt{-q}}\right] \right\}_0^{t-t_0}}$$

FORMS CONTAINING $(a + bx + cx^2)$

$X = a + bx + cx^2$ and $q = 4ac - b^2$

81. $\int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx + b}{\sqrt{q}}.$ $q > 0$

82. $\int \frac{dx}{X} = \frac{-2}{\sqrt{-q}} \tanh^{-1} \frac{2cx + b}{\sqrt{-q}}.$ $q < 0$

83. $\int \frac{dx}{X} = \frac{1}{\sqrt{-q}} \log \frac{2cx + b - \sqrt{-q}}{2cx + b + \sqrt{-q}}.$