$$= \frac{2}{n\pi} \int_0^{\pi} \cos n\left[\beta - \varepsilon \sin\beta\right] d\beta$$

Note that limits of integration $[0,\pi]$ on $\alpha \Rightarrow$ same limits on β. In the last paragraph we showed that $\beta(0)=0$, $\beta(\pi)=\pi$. We also know that β is monotonic.

$$= \frac{2}{n} \underbrace{\frac{1}{\pi} \int_0^{\pi} \cos\left[n\beta - (n\varepsilon)\sin\beta\right] d\beta}_{J_n(n\varepsilon)}$$

$J_n(n\varepsilon)$ (see Abramowicz & Stegun, Handbook of Mathematical Functions (AMS 55) 9.1.21 on p. 360.)

so

$$\beta - \alpha = 2 \sum_{n=1}^{\infty} \frac{1}{n} J_n(n\varepsilon) \sin n\alpha$$