Now just take the $\cos\omega\theta$ coefficient in this to find

$$(\omega_0^2-\omega^2) = \frac{3}{4}\lambda_3 a_1^2 - \frac{1}{6}\lambda_2^2 a_1^2 \frac{1}{\omega_0^2} + \lambda_2^2 \frac{1}{\omega_0^2} a_1^2$$

$$= \left(\frac{3}{4}\lambda_3 + \frac{5}{6}\frac{\lambda_2^2}{\omega_2^2}\right) a_1^2$$

Now the requirement that $\omega_0^2 - \omega^2 = 0$ gives the desired result,

$$\boxed{\frac{3}{4}\lambda_3 + \frac{5}{6}\frac{\lambda_2^2}{\omega_2^2} = 0}$$