Keep going. $(\omega_0^2-4\omega^2)a_2 = \frac{1}{2}\lambda_2 a_1^2 - \frac{1}{2}\lambda_4 a_1^4 + \cdots$

$$a_2 = -\frac{1}{3\omega_0^2}\left(\frac{1}{2}\lambda_2 a_1^2 - \frac{1}{2}\lambda_4 a_1^4 + \cdots\right)$$

$$(\omega_0^2 - 9\omega^2)a_3 = \frac{1}{4}\lambda_3 a_1^3 - \frac{5}{16}\lambda_5 a_1^5 + \cdots$$

So

$$a_0 = \frac{1}{\omega_0^2}\left(\frac{1}{2}\lambda_2 a_1^2 - \frac{3}{4}\lambda_4 a_1^4 + \cdots\right)$$

$$\omega^2 = \omega_0^2 - \frac{3}{4}\lambda_3 a_1^2 - \frac{10}{16}\lambda_5 a_1^4 + \cdots$$

$$a_2 = -\frac{1}{3\omega_0^2}\left(\frac{1}{2}\lambda_2 a_1^2 - \frac{1}{2}\lambda_4 a_1^4 + \cdots\right)$$

$$a_3 = -\frac{1}{8\omega_0^2}\left(\frac{1}{4}\lambda_3 a_1^3 - \frac{5}{16}\lambda_5 a_1^5 + \cdots\right)$$

Everything expressed in terms of a_1. Call this (1).

To 2nd order:

$$(\omega_0^2-\omega^2)a_1\cos\omega\theta + \sum_{m\neq 1}(\omega_0^2 - m^2\omega^2)a_n\cos m\omega\theta$$

$$= \sum_2^\infty \lambda_n\left[a_1\cos\omega\theta + \sum_{\ell\neq 1} a_\ell \cos \ell\omega\theta\right]^n$$

$$= \sum_2^\infty \lambda_n\left[a_1^n\cos^n\omega\theta + n a_1^{n-1}\cos^{n-1}\omega\theta \sum_{\ell\neq 1} a_\ell\cos\ell\omega\theta\right]$$ by the Binomial Thm.

$$\cos\omega\theta(\omega_0^2-\omega^2)a_1 = \left(\frac{3}{4}\lambda_3 a_1^3 + \frac{10}{16}\lambda_5 a_1^5\right)\cos\omega\theta + 2a_1\cos\omega\theta\left(-\frac{1}{3\omega_0^2}\right)\frac{1}{2}\lambda_2^2 a_1^2\cos 2\omega\theta + \cdots$$

(↑ using (1))

But $\cos\omega\theta\cos 2\omega\theta = \frac{1}{2}\left[\cos\omega\theta + \cos 3\omega\theta\right]$