Derivation of esoteric result:

Try a sol'n to I of the form $\epsilon = a_1 \cos\omega\theta + \sum_{m\neq 1} a_m \cos m\omega\theta$

We are going to require that $\omega^2 = \omega_0^2 \equiv 2-\nu$. But for now we will only assume that as $\lambda_n \to 0$ $n = 2,3,\ldots$ $\omega \to \omega_0$.

Hence as $\lambda_n \to 0$ $n=2,3,\ldots$ we see that $a_n \to 0$ $n = 2,3,\ldots$ since $\epsilon_0 = a_1 \cos\omega_0\theta$ is a solution to $\ddot{\epsilon} + (2-\nu)\epsilon = 0$.

Now get to work. In 1st order:

$$(\omega_0^2 - \omega^2) a_1 \cos\omega\theta + \sum_{m\neq 1} (\omega_0^2 - m^2\omega^2) a_m \cos m\omega\theta = \sum_2^\infty \lambda_n a_1^n \cos^n \omega\theta$$

$\cos^2\omega\theta = \frac{1}{2}(1+\cos 2\omega\theta)$, $\cos^3\omega\theta = \frac{1}{4}(\cos 3\omega\theta + 3\cos\omega\theta)$,

$\cos^4\omega\theta = \frac{1}{8}(\cos 4\omega\theta - 4\cos 2\omega\theta - 3)$, $\cos^5\omega\theta = \frac{1}{16}(\cos 5\omega\theta - 5\cos 3\omega\theta + 10\cos\omega\theta)$

← Now we equate coefficients of $\cos m\omega\theta$ for $m = 1,2,3,\ldots$

$$\Rightarrow \quad \boxed{\omega_0^2 a_0 = \frac{1}{2}\lambda_2 a_1^2 - \frac{3}{8}\lambda_4 a_1^4 + \ldots}$$

$$(\omega_0^2 - \omega^2) a_1 = \frac{3}{4}\lambda_3 a_1^3 + \frac{10}{16}\lambda_5 a_1^5 \quad \Rightarrow$$

$$\boxed{\omega^2 = \omega_0^2 - \frac{3}{4}\lambda_3 a_1^2 - \frac{10}{16}\lambda_5 a_1^4 + \ldots}$$

This shows in general how the nonlinear perturbations change the frequency. The simple pendulum is a good example. Its d.e. is $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$, which for small angles θ we approximate as $\ddot{\theta} + g/l\,\theta = 0$, with the result that ω_0 is independent of amplitude. But if we in[illegible] the nonlinear terms $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots$ then the frequency depends on amplitude!