53. Goldstein 2.24

Given $L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$, $I = \int_{t_1}^{t_2 = t_1 + 2\pi/\omega} L\,dt$. Make

the ansatz $x = \sum_{i=0}^{\infty} a_i \cos i\omega t \Rightarrow \dot{x} = -\sum_{i=0}^{\infty} i\omega a_i \sin i\omega t$

$$\Rightarrow L = \frac{m}{2}\sum_{ij} ij\,\omega^2 a_i a_j \sin i\omega t \sin j\omega t - \frac{k}{2}\sum_{ij} a_i a_j \cos i\omega t \cos j\omega t$$

and

$\int L\,dt$ contains terms of form $\int_{t_1}^{t_1+2\pi/\omega} dt \begin{Bmatrix} \sin i\omega t \sin j\omega t \\ \text{or} \\ \cos i\omega t \cos j\omega t \end{Bmatrix} = \delta_{ij}\frac{\pi}{\omega}$.

Therefore the integral gives

$$\boxed{I(a_0\, a_1 \cdots) = \frac{\pi}{\omega}\sum_{j=0}^{\infty}\left(\frac{m}{2} j^2\omega^2 - k/2\right) a_j^2}$$

For an extremum must have $\partial I/\partial a_i = 0 \;\forall i$.

But $\frac{\partial I}{\partial a_i} = 0 \Rightarrow \left(\frac{m}{2} i^2\omega^2 - k/2\right) a_i = 0 \Rightarrow$

$$a_i = 0 \quad \text{or} \quad \left(\frac{m}{2} i^2 \omega^2 - k/2\right) = 0$$

This is satisfied by $\boxed{a_i = 0 \quad i \neq 1}$ and

$\left(\frac{m}{2} i^2\omega^2 - k/2\right) = 0$ for $i = 1 \Rightarrow \boxed{\omega^2 = k/m} \Rightarrow$

$\boxed{x = a_1 \cos \omega_0 t \quad \text{with } \omega_0^2 = k/m}$

However could also have $a_i = 0 \quad i \neq j$

and $\left(\frac{m}{2} i^2\omega^2 - k/2\right) = 0$ for $i = j \Rightarrow \omega^2 = \frac{1}{j^2}\frac{k}{m}$

$\Rightarrow \boxed{x = a_j \cos j\omega t = a_j \cos j\sqrt{\frac{k}{j^2 m}}\, t = a_j \cos \omega_0 t \quad \text{which is same as before.}}$