52. Goldstein 2.17

Given $T = \sum_i f_i(q_i)\,\dot{q}_i^{\,2}$ and $V = \sum_i V_i(q_i)$, then

$$L = T - V = \sum_i L_i(q_i, \dot{q}_i) \quad \text{where}$$

$$L_i = f_i(q_i)\,\dot{q}_i^{\,2} - V_i(q_i).$$ L is separable.

Lagrange's eqns give $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \Rightarrow$

$$\frac{d}{dt}\frac{\partial L_i(q_i,\dot{q}_i)}{\partial \dot{q}_i} - \frac{\partial L_i(q_i,\dot{q}_i)}{\partial q_i} = 0.$$

∴ Eqns of motion for different q_i are uncoupled.

Same result as if L_i were the Lagrangian!

Since $\frac{\partial L_i}{\partial t} = 0$, we have $H_i = \frac{\partial L_i}{\partial \dot{q}_i}\dot{q}_i - L_i = \text{cnst} = c_i.$

But $\frac{\partial L_i}{\partial \dot{q}_i} = 2 f_i(q_i)\,\dot{q}_i \Rightarrow H_i = f_i(q_i)\,\dot{q}_i^{\,2} + V_i(q_i) = c_i$

$$\therefore \quad \dot{q}_i^{\,2} = [c_i - V_i(q_i)]/f_i(q_i) = (dq_i/dt)^2.$$

$$\therefore \quad \boxed{t - t^\circ = \int_{q_i^\circ}^{q_i} d\bar{q}_i \left\{ \frac{f_i(\bar{q}_i)}{[c_i - V_i(\bar{q}_i)]} \right\}^{1/2}}$$

solution by quadrature