51

Goldstein 2.1

G2.3 Brachistochrone Problem.

For convenience define the $+y$ axis in the direction of $\vec{g}$. Then $\frac{1}{2}mv^2 = mgy$, hence

$$v = (2gy)^{1/2}$$

$$t_{AB} = \int_A^B dt = \int \frac{dt}{ds}\frac{ds}{dx}\,dx = \int \sqrt{\frac{1+y'^2}{2gy}}\,dx$$

$$\delta t_{AB} = 0 \quad \Rightarrow \quad \delta \int \sqrt{\frac{1+y'^2}{y}}\,dx = 0$$

Now as in G2.2, we set $f = \sqrt{\frac{1+y'^2}{y}}$ and get

$$y'\frac{\partial f}{\partial y'} - f = \text{const} = c_1$$

$$\frac{y'^2}{\sqrt{y(1+y'^2)}} - \sqrt{\frac{1+y'^2}{y}} = c_1$$

$$y'^2 - (1+y'^2) = c_1\sqrt{y(1+y'^2)}$$

$$\Rightarrow \quad y(1+y'^2) = c_2 = \text{const} > 0.$$

$$y' = \sqrt{\frac{c_2 - y}{y}} \qquad dx = dy\sqrt{\frac{y}{c_2 - y}} \qquad \underline{c_2 > 0}$$

Note: This is one of those DE's good only until $y = c_2$.