$$\frac{d\varphi}{d\theta} = \frac{\sin\alpha}{\sin\theta}\left(\sin^2\theta - \sin^2\alpha\right)^{-1/2}$$

$$\varphi = \sin\alpha \int \frac{d\theta}{\sin\theta\sqrt{\sin^2\theta - \sin^2\alpha}}$$

$$= \sin\alpha \int \frac{d\theta \csc^2\theta}{\sqrt{1 - \sin^2\alpha\csc^2\theta}}$$

But $d(\operatorname{ctn}\theta) = -\csc^2\theta\, d\theta$

and $1 + \operatorname{ctn}^2\theta = \csc^2\theta$

So

$$\varphi = -\sin\alpha \int \frac{d(\operatorname{ctn}\theta)}{\sqrt{1 - \sin^2\alpha - \sin^2\alpha\operatorname{ctn}^2\theta}}$$

$$= -\int \frac{d(\operatorname{ctn}\theta)}{\sqrt{\operatorname{ctn}^2\alpha - \operatorname{ctn}^2\theta}}$$

$$= -\int \frac{d(\tan\alpha\operatorname{ctn}\theta)}{\sqrt{1-(\tan\alpha\operatorname{ctn}\theta)^2}} = -\sin^{-1}(\tan\alpha\operatorname{ctn}\theta) - \beta$$

β a constant

$\therefore \sin(\varphi + \beta) = -\tan\alpha\operatorname{ctn}\theta$

$\therefore \sin\theta(\sin\varphi\cos\beta + \cos\varphi\sin\beta) + \tan\alpha\cos\theta = 0$

Now, on a unit sphere:

$x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, $z = \cos\theta$, so

$\Rightarrow$ $\boxed{y\cos\beta + x\sin\beta + z\tan\alpha = 0}$ is equation of curve.

Define $\vec{n} = \sin\beta\,\hat{e}_x + \cos\beta\,\hat{e}_y + \tan\alpha\,\hat{e}_z$

Then $\boxed{\vec{r}\cdot\vec{n} = 0}$ is equation of curve.

Since $\vec{r}\cdot\vec{n} = 0$

$\vec{r}$ must describe a plane perpendicular to the vector $\vec{n}$

This plane passes thru the center of the sphere, since $\vec{r} = (0,0,0)$ satisfies $\vec{r}\cdot\vec{n} = 0$

So the shortest distance between two points on the surface of a sphere is the intersection of the sphere with the plane containing A, B, and the centers But this is the definition of a great circle.