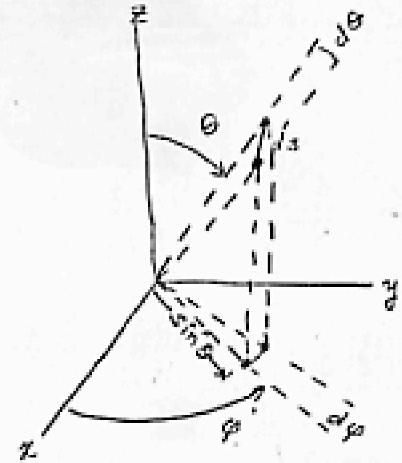
Goldstein 2.4 cont

Alternate solution

using θ as an independent variable.



ds = element of length on the surface of a sphere of radius 1.

So $ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

Given 2 points A and B, the distance between them is $s = \int_A^B ds$

$$s = \int_{\theta_A}^{\theta_B} d\theta \sqrt{1+\sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2} = \int_{\theta_A}^{\theta_B} f\left(\frac{d\varphi}{d\theta}, \theta\right) d\theta$$

So $f = \left[1 + \sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2\right]^{1/2}$.

Euler-Lagrange gives

$$\frac{\partial f}{\partial \varphi} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial\left(\frac{d\varphi}{d\theta}\right)}\right) = 0$$

$$\Rightarrow \frac{d}{d\theta}\left\{\left[1+\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2\right]^{-\frac{1}{2}} \sin^2\theta \frac{d\varphi}{d\theta}\right\} = 0$$

$$\Rightarrow \left[1+\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2\right]^{-\frac{1}{2}} \sin^2\theta \frac{d\varphi}{d\theta} = \text{constant} = \sin\alpha$$

$$\sin^4\theta \left(\frac{d\varphi}{d\theta}\right)^2 = \sin^2\alpha \left[1+\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2\right]$$

$$\left(\frac{d\varphi}{d\theta}\right)^2 \sin^2\theta \left(\sin^2\theta - \sin^2\alpha\right) = \sin^2\alpha$$