$\boxed{\text{Goldstein 2.4.}}$

Show that geodesics on spheres are great circles.

$ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\varphi^2$ So we must extremize

$$J = a\int_0^{2\pi} f\left(\theta, \frac{d\theta}{d\varphi}\right) d\varphi \quad \text{with } f\left(\theta, \frac{d\theta}{d\varphi}\right) = \left[\left(\frac{d\theta}{d\varphi}\right)^2 + \sin^2\theta\right]^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial \theta'} = \frac{\theta'}{\sqrt{\theta'^2 + \sin^2\theta}}$$

$$\frac{\partial f}{\partial \theta} = \frac{\sin\theta\cos\theta}{\sqrt{\theta'^2 + \sin^2\theta}}$$

so the Euler-Lagrange eqn. is:

$$\frac{d}{d\varphi}\frac{\partial f}{\partial \theta'} - \frac{\partial f}{\partial \theta} = 0 \quad \text{or}$$

$$\frac{\partial^2 f}{\partial\theta\, d\theta'}\theta' + \frac{\partial^2 f}{\partial \theta'^2}\theta'' - \frac{\partial f}{\partial \theta} = 0.$$

$$\frac{\partial^2 f}{\partial\theta\,\partial\theta'}\theta'^2 + \frac{\partial^2 f}{\partial\theta'^2}\theta'\theta'' - \theta'\frac{\partial f}{\partial\theta} = 0$$

$$\frac{d}{d\varphi}\left[\theta'\frac{\partial f}{\partial\theta'} - f\right] = 0 \quad \text{or}$$

$$\theta'\frac{\partial f}{\partial \theta'} - f = \text{const} \doteq -\frac{1}{K}$$

$$\frac{\theta'^2}{\sqrt{\theta'^2 + \sin^2\theta}} - \sqrt{\theta'^2 + \sin^2\theta} = -\frac{1}{K}$$