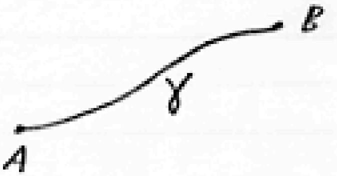
Goldstein 2.3. Prove by Calculus of Variations that the shortest distance between 2 points in 3 space is a straight line.



$$S = \int_A^B dS = \int_{x_A}^{x_B} \underbrace{\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2}}_{\mathcal{L}}\, dx$$

So $\mathcal{L} = \sqrt{1+y'^2+z'^2}$

$$\frac{\partial \mathcal{L}}{\partial y'} = \frac{y'}{\mathcal{L}} \quad , \quad \frac{\partial \mathcal{L}}{\partial y} = 0$$

$$\frac{\partial \mathcal{L}}{\partial z'} = \frac{z'}{\mathcal{L}} \quad , \quad \frac{\partial \mathcal{L}}{\partial z} = 0$$

So

$$y' = \alpha \mathcal{L} \quad , \quad z' = \beta \mathcal{L} \qquad \alpha, \beta \text{ const}^{\underline{s}}$$

So $\mathcal{L} = \left[1+\alpha^2\mathcal{L}^2+\beta^2\mathcal{L}^2\right]^{\frac{1}{2}}$

$$\mathcal{L} = \left[1 + (\alpha^2+\beta^2)\mathcal{L}^2\right]^{\frac{1}{2}}$$

Furthermore $\alpha^2+\beta^2 < 1$ since $\alpha^2+\beta^2 \geq 1 \Rightarrow$

$$\mathcal{L} \geq \left[1+\mathcal{L}^2\right]^{\frac{1}{2}} > \mathcal{L} \text{, a contradiction.}$$

So squaring $\mathcal{L}^2 - (\alpha^2+\beta^2)\mathcal{L}^2 = 1$

$$\mathcal{L}^2 = \frac{1}{1-(\alpha^2+\beta^2)}$$

ok since $\alpha^2+\beta^2 < 1$

So $\mathcal{L}$ is a constant and $y' = \text{const}$, $z' = \text{const} \Rightarrow$

$$y = c_1 x + d_1 \qquad z = c_2 x + d_2$$

QED