Consider the differential form $\sum_{b=1}^{m} C_b(z)\,dz_b$ where the $C_b(z)$ are specified functions of the variables $z_1, z_2, \dots, z_m$. A differential form is called "<u>exact</u>" or "<u>perfect</u>" if there exists a function f such that $df = \sum_{b=1}^{m} C_b(z)\,dz_b$.

Theorem: $\partial C_b/\partial z_a - \partial C_a/\partial z_b = 0 \;\forall a,b \Leftrightarrow \exists\, f(z)$ such that $df = \sum_b C_b(z)\,dz_b$.

Necessary and Sufficient Condition for a Perfect Differential

<u>Proof.</u>

1) $\Leftarrow$: $df = \sum_b \partial f/\partial z_b\, dz_b \rightarrow C_b(z) = \partial f/\partial z_b \rightarrow \partial C_b/\partial z_a - \partial C_a/\partial z_b = \dfrac{\partial^2 f}{\partial z_a \partial z_b} - \dfrac{\partial^2 f}{\partial z_b \partial z_a} = 0$

2) $\Rightarrow$: Suppose $\partial C_b/\partial z_a - \partial C_a/\partial z_b = 0$. Let z^0 + z' be two arbitrary points, and let P be some path joining them. Consider the integral

$$I[P] = \int_{z^0}^{z'} \sum_b C_b(z)\,dz_b \quad \text{or} \quad I[z(t)] = \int_{t^0}^{t'} \Big\{ \sum_b C_b(z)\,\dot z_b \Big\}\,d\tau$$

where $z(t)$ is a parameterization of the path and $\dot z_b = dz_b/d\tau$.

Define a "Lagrangian" $\mathcal{L}$ by $\mathcal{L}(z,\dot z) = \sum_b C_b(z)\,\dot z_b$. Then by the calculus of variations,

$$\delta I = \int_{t^0}^{t'} dt \Big\{ \sum_a \Big(-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot z_a} + \frac{\partial \mathcal{L}}{\partial z_a} \Big) \delta z_a \Big\}$$

for a varied path with the same end points.

But, $\partial \mathcal{L}/\partial \dot z_a = C_a(z)$, $\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot z_a} = \sum_b \dfrac{\partial C_a}{\partial z_b}\dot z_b$, $\dfrac{\partial \mathcal{L}}{\partial z_a} = \sum_b \dfrac{\partial C_b}{\partial z_a}\dot z_b$. Therefore,

$$\Big(-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot z_a} + \frac{\partial \mathcal{L}}{\partial z_a} \Big) = \sum_b \Big(\frac{\partial C_b}{\partial z_a} - \frac{\partial C_a}{\partial z_b} \Big) \dot z_b = 0$$

since $\partial C_b/\partial z_a - \partial C_a/\partial z_b = 0$ by assumption.

Thus, $\delta I = 0$ for $\forall\, \delta z_a$, and the integral I is <u>path independent</u>! Now define $f(z)$ by

$$f(z) = \int_{z^0}^{z} \sum_b C_b(z')\,dz'_b .$$

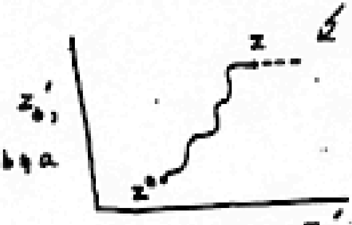
The integral is well defined, and depends only on the end points, because it is path independent.

Then, by a suitable choice of path,

$\partial f/\partial z_a = C_a(z)$, and hence

$$df = \sum_b \partial f/\partial z_b\, dz_b = \sum_b C_b(z)\,dz_b .$$

See picture for a suitable path for computing $\partial f/\partial z_a$.



At end of path, only z'_a varies; and all other z'_b with $b \neq a$ are held fixed.

Suitable Path for Computing $\partial f/\partial z_a$