$\mathcal{L}_1$ and $\mathcal{L}_2$ do not differ by a total time 1/1

40. derivative, for if they did we would have a relation of the form $\mathcal{L}_2 - \mathcal{L}_1 = \frac{d}{dt} f(q,t) = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t}$, which is manifestly not the case.

$\mathcal{L}_1$ is of course the usual Lagrangian for the harmonic oscillator with unit frequency:

$$\mathcal{L}_1 = \tfrac{1}{2}(\dot{q}^2 - q^2)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q} \qquad \frac{\partial \mathcal{L}}{\partial q} = -q$$

$$0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \ddot{q} + q$$

Now consider $\mathcal{L}_2 = \left[\tfrac{1}{2} q\dot{q}^2 - \tfrac{1}{3} q^3\right]\sin t + \tfrac{1}{6}\dot{q}^3 \cos t$

$$\frac{\partial \mathcal{L}_2}{\partial \dot{q}} = q\dot{q}\sin t + \tfrac{1}{2}\dot{q}^2 \cos t$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}_2}{\partial \dot{q}} = \dot{q}^2 \sin t + q\ddot{q}\sin t + q\dot{q}\cos t + \dot{q}\ddot{q}\cos t - \tfrac{1}{2}\dot{q}^2 \sin t$$

$$\frac{\partial \mathcal{L}_2}{\partial q} = \tfrac{1}{2}\dot{q}^2 \sin t - q^2 \sin t$$

So

$$\frac{d}{dt}\frac{\partial \mathcal{L}_2}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = q\ddot{q}\sin t + \dot{q}\ddot{q}\cos t + q\dot{q}\cos t + q^2 \sin t$$

$$= (q\sin t + \dot{q}\cos t)(\ddot{q} + q) = 0$$

If the first factor is non zero, then $\ddot{q} + q = 0$.

If the first factor is zero for some interval in t, then differentiating we get $(\ddot{q} + q)\cos t = 0$.

So, in any case, $\ddot{q} + q = 0$.