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As a last example, we consider a system in which there are moving constraints. A bead of mass m slides without friction on a circular hoop of radius a. The hoop lies in a vertical plane which is constrained to rotate about a vertical diameter with constant angular velocity ω. There is just one degree of freedom, and inasmuch as we are not interested in the forces of constraint, we choose a single coordinate θ which measures the angle around the circle from the bottom of the vertical diameter to the bead (Fig. 9.9). The kinetic energy is then

$$T = \tfrac{1}{2}ma^2\dot{\theta}^2 + \tfrac{1}{2}ma^2\omega^2 \sin^2\theta, \tag{9.147}$$

and the potential energy is

$$V = -mga\cos\theta. \tag{9.148}$$

The Lagrangian function is

(a)

$$L = \tfrac{1}{2}ma^2\dot{\theta}^2 + \tfrac{1}{2}ma^2\omega^2 \sin^2\theta + mga\cos\theta. \tag{9.149}$$

The Lagrange equation of motion can easily be written out, but this is unnecessary, for we notice that

$$\frac{\partial L}{dt} = 0,$$

and therefore, by Eq. (9.126), the quantity

$$\mathcal{H} = \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} - L = \tfrac{1}{2}ma^2\dot{\theta}^2 - \tfrac{1}{2}ma^2\omega^2\sin^2\theta - mga\cos\theta = \text{`}E\text{'} \tag{9.150}$$

is constant. The constant 'E' is not the total energy $T + V$, for the middle term has the wrong sign. The total energy is evidently not constant in this case. (What

Fig. 9.9 A bead sliding on a rotating hoop.

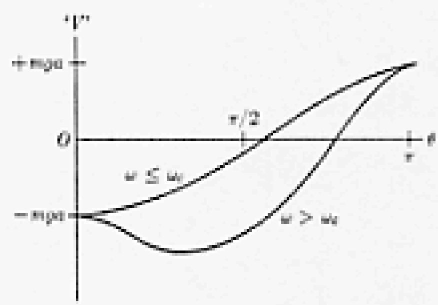

Fig. 9.10 Effective potential energy for system shown in Fig. 9.9.

force does the work which produces changes in $T + V$?) We may note, however, that we can interpret Eq. (9.149) as a Lagrangian function in terms of a fixed coordinate system with the middle term regarded as part of an effective potential energy:

$$\text{`}V\text{'}(\theta) = -\tfrac{1}{2}ma^2\omega^2\sin^2\theta - mga\cos\theta. \tag{9.151}$$

The energy according to this interpretation is 'E'. The first term in 'V'(θ) is the potential energy associated with the centrifugal force which must be added if we regard the rotating system as fixed. The effective potential is plotted in Fig. 9.10. The shape of the potential curve depends on whether ω is greater or less than a critical angular velocity

$$\omega_c = (g/a)^{1/2}. \tag{9.152}$$

It is left to the reader to show this, and to discuss the nature of the motion of the bead in the two cases.