b) Now find the Hamiltonian.

$$\begin{cases} p_\rho = \dfrac{\partial \mathcal{L}}{\partial \dot\rho} = m(1+4\rho^2/a^2)\dot\rho \\ p_\phi = \dfrac{\partial \mathcal{L}}{\partial \dot\phi} = m\rho^2\dot\phi \end{cases}$$

$$\Rightarrow \begin{cases} \dot\rho = p_\rho / [m(1+4\rho^2/a^2)] \\ \dot\phi = p_\phi / (m\rho^2) \end{cases}$$

$$\mathcal{H} = p_\rho \dot\rho + p_\phi \dot\phi - \mathcal{L}$$

$$= m(1+4\rho^2/a^2)\dot\rho^2 + m\rho^2\dot\phi^2 - \tfrac{1}{2}m\left[(1+4\rho^2/a^2)\dot\rho^2 + \rho^2\dot\phi^2\right] + mg\rho^2/a$$

$$= \tfrac{1}{2}m\left[(1+4\rho^2/a^2)\dot\rho^2 + \rho^2\dot\phi^2\right] + mg\rho^2/a$$

or

$$\boxed{\mathcal{H} = \left[\frac{p_\rho^2}{2m(1+4\rho^2/a^2)} + \frac{p_\phi^2}{2m\rho^2}\right] + mg\rho^2/a}$$